Relational to assertional abstraction

From a relation between initial and current state to an assertion on current states.

$$\langle \wp(\mathcal{X}\times\mathcal{Y}),\ \subseteq\rangle \xrightleftharpoons[\alpha_{\downarrow 2}]{\gamma_{\downarrow 2}} \langle \wp(\mathcal{Y}),\ \subseteq\rangle,\ \langle \wp(\wp(\mathcal{X}\times\mathcal{Y})\times\wp(\mathcal{X}\times\mathcal{Z})),\ \subseteq\rangle \xrightleftharpoons[\dot{\alpha}_{\downarrow 2}]{\dot{\gamma}_{\downarrow 2}} \langle \wp(\wp(\mathcal{Y})\times\wp(\mathcal{Z})),\ \subseteq\rangle \quad (23)$$

$$\alpha_{\downarrow^2}(P) \triangleq \{\sigma \mid \exists \sigma_0 \,.\, \langle \sigma_0,\ \sigma\rangle \in P\} \qquad \gamma_{\downarrow^2}(Q) \triangleq \mathcal{X}\times Q \quad (24)$$

$$\dot{\alpha}_{\downarrow^2}(R) \triangleq \{\langle \alpha_{\downarrow^2}(P),\ \alpha_{\downarrow^2}(Q)\rangle \mid \langle P,\ Q\rangle \in R\}$$

$$\dot{\gamma}_{\downarrow^2}(R') \triangleq \{\langle P, Q\rangle \mid \langle \alpha_{\downarrow}^2(P), \alpha_{\downarrow}^2(Q)\rangle \in R'\}$$ (1)

← project on second component

(1) $\dot{\gamma}_{\downarrow^2}$ is wrong in POPL24, as discovered by Florian Sextl from TU Wien (email of May 27, 2024)

Proof of (24).

- $\alpha_{\downarrow}2(P) \subseteq Q$

$\Leftrightarrow \{\sigma \mid \exists \sigma_0 . \langle \sigma_0, \sigma\rangle \in P\} \subseteq Q$ $\quad$ $\{$def. $\alpha_{\downarrow}^2\}$

$\Leftrightarrow \forall \sigma . (\exists \sigma_0 (\sigma_0, \sigma) \in P) \Rightarrow \sigma \in Q$ $\quad$ $\{$def. $\subseteq\}$

$\Leftrightarrow \forall \sigma . \forall \sigma_0 . \langle \sigma_0, \sigma\rangle \in P \Rightarrow \sigma \in Q$ $\quad$ $\{$def. $\Rightarrow, \exists\}$

$\Leftrightarrow \forall \sigma_0 . \forall \sigma . \langle \sigma_0, \sigma\rangle \in P \Rightarrow \langle \sigma_0, \sigma\rangle \in \mathcal{X} \times Q$ $\quad$ $\{\sigma_0 \in \mathcal{X}\}$

$\Leftrightarrow P \subseteq \mathcal{X} \times Q \triangleq \gamma_{\downarrow}^2(Q)$ $\quad$ $\{$def. $\subseteq\}$

- $\dot{\alpha}_{\downarrow}2(R) \subseteq R'$

$\Leftrightarrow \{\langle \alpha_{\downarrow}^2(P), \alpha_{\downarrow}^2(Q)\rangle \mid \langle P, Q\rangle \in R\} \subseteq R'$ $\quad$ $\{$def. $\dot{\alpha}_{\downarrow}^2\}$

$\Leftrightarrow \forall \langle P, Q\rangle \in R . \langle \alpha_{\downarrow}^2(P), \alpha_{\downarrow}^2(Q)\rangle \in R'$ $\quad$ $\{$def $\subseteq\}$

$\Leftrightarrow R \subseteq \{\langle P, Q\rangle \mid \langle \alpha_{\downarrow}^2(P), \alpha_{\downarrow}^2(Q)\rangle \in R'\}$ $\quad$ $\{$def $\subseteq\}$

$\Leftrightarrow R \subseteq \dot{\gamma}_{\downarrow}^2(R')$ $\quad$ $\{$def. $\dot{\gamma}_{\downarrow}^2\}$