« Proving Program Invariance and Termination by Parametric Abstraction, Lagrangian Relaxation and Semidefinite Programming »

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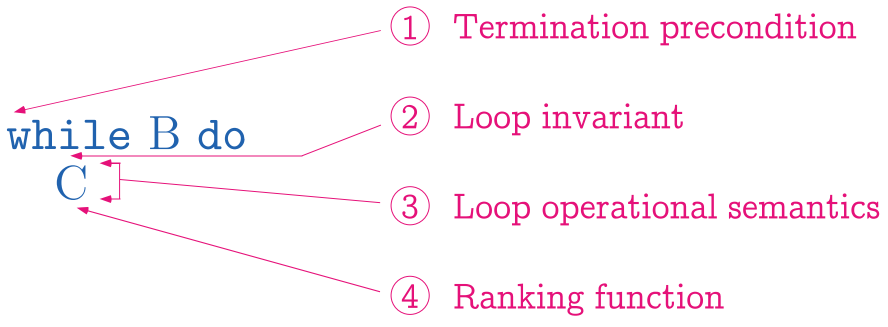
Overview of the Termination Analysis Method

Proving Termination of a Loop



The main point in this talk is (4).

Proving Termination of a Loop

1. Perform an iterated forward/backward relational static analysis of the loop with *termination hypothesis* to determine a *necessary* proper termination precondition
2. Assuming the *termination precondition*, perform an forward relational static analysis of the loop to determine the loop invariant
3. Assuming the loop invariant, perform an forward relational static analysis of the loop body to determine the loop abstract operational semantics
4. Assuming the loop semantics, use an abstraction of Floyd's ranking function method to prove termination of the loop

Arithmetic Mean Example

```
while (x <> y) do
    x := x - 1;
    y := y + 1
od
```

The polyhedral abstraction used for the static analysis of the examples is implemented using Bertrand Jeannet's NewPolka library.

Arithmetic Mean Example

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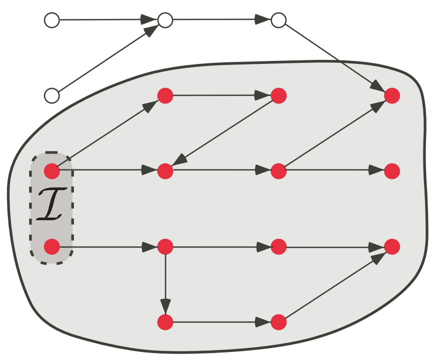
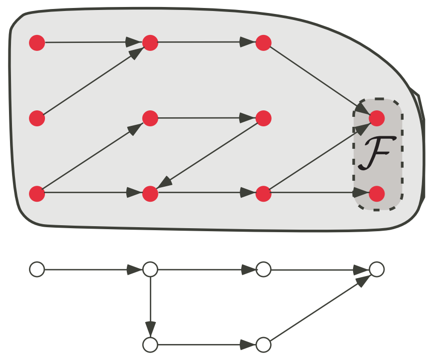
Forward/reachability properties



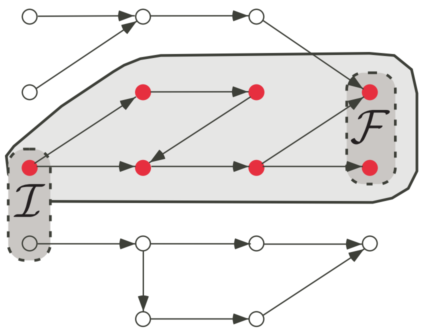
Example: partial correctness (must stay into safe states)

Backward/ancestry properties



Example: termination (must reach final states)

Forward/backward properties



Example: total correctness (stay safe while reaching final states)

Principle of the iterated forward/backward iteration-based approximate analysis

– Overapproximate

$$\mathsf{lfp}\ F \sqcap \mathsf{lfp}\ B$$

by overapproximations of the decreasing sequence

$$\begin{aligned} X^0 &= \top \\ &\ldots \\ X^{2n+1} &= \mathsf{lfp}\ \lambda Y \cdot X^{2n} \sqcap F(Y) \\ X^{2n+2} &= \mathsf{lfp}\ \lambda Y \cdot X^{2n+1} \sqcap B(Y) \\ &\ldots \end{aligned}$$

Arithmetic Mean Example: Termination Precondition (1)

```
{x>=y}
  while (x <> y) do
    {x>=y+2}
      x := x - 1;
    {x>=y+1}
      y := y + 1
    {x>=y}
  od
{x=y}
```


Idea 1

The auxiliary termination counter method

Arithmetic Mean Example: Termination Precondition (2)

```
{x=y+2k,x>=y}
  while (x <> y) do
    {x=y+2k,x>=y+2}
      k := k - 1;
    {x=y+2k+2,x>=y+2}
      x := x - 1;
    {x=y+2k+1,x>=y+1}
      y := y + 1
    {x=y+2k,x>=y}
  od
{x=y,k=0}
  assume (k = 0)
{x=y,k=0}
```

Add an auxiliary termination counter to enforce (bounded) termination in the backward analysis!

Arithmetic Mean Example: Loop Invariant

```
  assume ((x=y+2*k) & (x>=y));
{x=y+2k,x>=y}
  while (x <> y) do
    {x=y+2k,x>=y+2}
      k := k - 1;
    {x=y+2k+2,x>=y+2}
      x := x - 1;
    {x=y+2k+1,x>=y+1}
      y := y + 1
    {x=y+2k,x>=y}
  od
{k=0,x=y}
```


Arithmetic Mean Example

1. Perform an iterated forward/backward relational static analysis of the loop with *termination hypothesis* to determine a *necessary* proper termination precondition
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Arithmetic Mean Example

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Arithmetic Mean Example: Body Relational Semantics

Case x < y:

```
assume (x=y+2*k)&(x>=y+2);
{x=y+2k,x>=y+2}
assume (x < y);
empty(6)
assume (x0=x)&(y0=y)&(k0=k);
empty(6)
k := k - 1;
x := x - 1;
y := y + 1
empty(6)
```

Case x > y:

```
assume (x=y+2*k)&(x>=y+2);
{x=y+2k,x>=y+2}
assume (x > y);
{x=y+2k,x>=y+2}
assume (x0=x)&(y0=y)&(k0=k);
{x=y+2k0,y=y0,x=x0,x=y+2k,
                        x>=y+2}
k := k - 1;
x := x - 1;
y := y + 1
{x+2=y+2k0,y=y0+1,x+1=x0,
                   x=y+2k,x>=y}
```


Floyd's method for termination of `while B do C`

Given a loop invariant I, find an $\mathbb{R}/\mathbb{Q}/\mathbb{Z}$-valued unkown rank function r such that:

– The rank is *nonnegative*:

$$\forall\, x_0, x : I(x_0) \wedge [\![\mathtt{B};\mathtt{C}]\!](x_0, x) \Rightarrow r(x_0) \geq 0$$

– The rank is *strictly decreasing*:

$$\forall\, x_0, x : I(x_0) \wedge [\![\mathtt{B};\mathtt{C}]\!](x_0, x) \Rightarrow r(x) \leq r(x_0) - \eta$$

$\eta \geq 1$ for $\mathbb{Z}$, $\eta > 0$ for $\mathbb{R}/\mathbb{Q}$ to avoid Zeno $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\ldots$

Arithmetic Mean Example

1. Perform an iterated forward/backward relational static analysis of the loop with *termination hypothesis* to determine a *necessary* proper termination precondition
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Arithmetic Mean Example: Ranking Function

```
» clear all;
[v0,v] = variables('x','y','k')
% linear inequalities
%       x0 y0 k0
Ai = [  0  0  0];
%        x  y  k
Ai_ = [  1 -1  0]; % x0 - y0 >= 0
bi = [0];
[N Mk(:,:,:)]=linToMk(Ai,Ai_,bi);
% linear equalities
%       x0 y0 k0
Ae =  [  0  0 -2;
         0 -1  0;
        -1  0  0;
         0  0  0];
%        x  y  k
Ae_ = [  1 -1  0;  % x - y - 2*k0 - 2 = 0
         0  1  0;  % y - y0 - 1 = 0
         1  0  0;  % x - x0 + 1 = 0
         1 -1 -2]; % x - y - 2*k = 0
be = [2; -1; 1; 0];
[M Mk(:,:,N+1:N+M)]=linToMk(Ae,Ae_,be);
```

Input the loop abstract semantics

```
» display_Mk(Mk, N, v0, v);
...

 +1.x -1.y >= 0
 -2.k0 +1.x -1.y +2 = 0
 -1.y0 +1.y -1 = 0
 -1.x0 +1.x +1 = 0
 +1.x -1.y -2.k = 0
...
```

- Display the abstract semantics of the loop `while B do C`
- compute ranking function, if any

```
» [diagnostic,R] = termination(v0, v, Mk, N, 'integer', 'linear');
» disp(diagnostic)
   feasible (bnb)
» intrank(R, v)
```

`r(x,y,k) = +4.k -2`

Proving Termination by Parametric Abstraction, Lagrangian Relaxation and Semidefinite Programming

Idea 2

Express the loop invariant and relational semantics as numerical positivity constraints

Relational semantics of `while B do C od` loops

- $x_0 \in \mathbb{R}/\mathbb{Q}/\mathbb{Z}$: values of the loop variables *before* a loop iteration
- $x \in \mathbb{R}/\mathbb{Q}/\mathbb{Z}$: values of the loop variables *after* a loop iteration
- $I(x_0)$: loop invariant, $[\![\mathtt{B};\mathtt{C}]\!](x_0, x)$: relational semantics of *one iteration of the loop body*
- $I(x_0) \wedge [\![\mathtt{B};\mathtt{C}]\!](x_0, x) = \bigwedge_{i=1}^{N} \sigma_i(x_0, x) \geqslant_i 0 \quad (\geqslant_i \in \{>, \geq, =\})$
- not a restriction for numerical programs

Example of linear program (Arithmetic mean)

$$[A\;A'][x_0\;x]^\top \geqslant b$$

```
{x=y+2k,x>=y}
while (x <> y) do
    k := k - 1;
    x := x - 1;
    y := y + 1
od
```

```
+1.x -1.y >= 0
-2.k0 +1.x -1.y +2 = 0
-1.y0 +1.y -1 = 0
-1.x0 +1.x +1 = 0
+1.x -1.y -2.k = 0
```

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -2 \end{array}\right] \begin{bmatrix} x_0 \\ y_0 \\ k_0 \\ x \\ y \\ k \end{bmatrix} \begin{array}{c} \geq \\ = \\ = \\ = \\ = \end{array} \begin{bmatrix} 0 \\ -2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Example of quadratic form program (factorial)

$$[x\;x']A[x\;x']^\top + 2[x\;x']\,q + r \geqslant 0$$

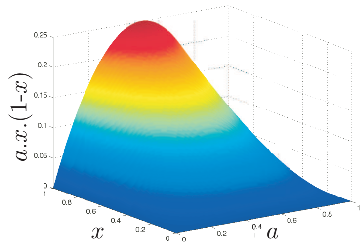
```
n := 0;
f := 1;
while (f <= N) do
    n := n + 1;
    f := n * f
od
```

```
-1.f0 +1.N0 >= 0
+1.n0 >= 0
+1.f0 -1 >= 0
-1.n0 +1.n -1 = 0
+1.N0 -1.N = 0
-1.f0.n +1.f = 0
```

$$[n_0 f_0 N_0 n f N] \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_0 \\ f_0 \\ N_0 \\ n \\ f \\ N \end{bmatrix} + 2[n_0 f_0 N_0 n f N] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + 0 = 0$$

Example of semialgebraic program (logistic map)

```
eps = 1.0e-9;
while (0 <= a) & (a <= 1 - eps)
        & (eps <= x) & (x <= 1) do
   x := a*x*(1-x)
od
```



Floyd's method for termination of `while B do C`

Find an $\mathbb{R}/\mathbb{Q}/\mathbb{Z}$-valued unkown rank function r and $\eta > 0$ such that:

– The rank is *nonnegative*:

$$\forall\, x_0, x : \bigwedge_{i=1}^{N} \sigma_i(x_0, x) \geqslant_i 0 \Rightarrow r(x_0) \geq 0$$

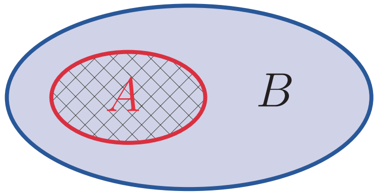
– The rank is *strictly decreasing*:

$$\forall\, x_0, x : \bigwedge_{i=1}^{N} \sigma_i(x_0, x) \geqslant_i 0 \Rightarrow r(x_0) - r(x) - \eta \geq 0$$

Idea 3

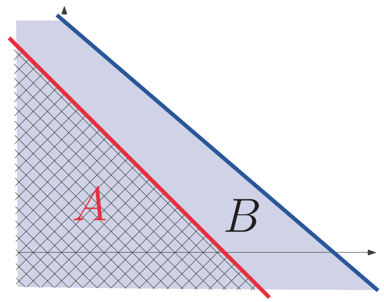
Eliminate the conjunction $\bigwedge$ and implication $\Rightarrow$ by Lagrangian relaxation

Implication (general case)



$$A \Rightarrow B$$
$$\Leftrightarrow$$
$$\forall x \in A : x \in B$$

Implication (linear case)



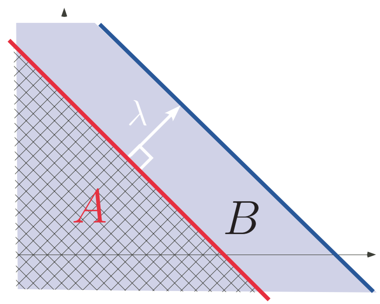
$A \Rightarrow B$ (assuming $A \neq \emptyset$)

$\Leftarrow$ (soundness)

$\Rightarrow$ (completeness)

border of A parallel to border of B

Lagrangian relaxation (linear case)



Lagrangian relaxation, formally

Let $\mathbb{V}$ be a finite dimensional linear vector space, $N > 0$ and $\forall k \in [0, N] : \sigma_k \in \mathbb{V} \mapsto \mathbb{R}$.

$$\forall x \in \mathbb{V} : \left(\bigwedge_{k=1}^{N} \sigma_k(x) \geq 0 \right) \Rightarrow (\sigma_0(x) \geq 0)$$

$\Leftarrow$ soundness (Lagrange)

$\Rightarrow$ completeness (*lossless*)

$\not\Rightarrow$ incompleteness (*lossy*)

$$\exists \lambda \in [1, N] \mapsto \mathbb{R}^+ : \forall x \in \mathbb{V} : \sigma_0(x) - \sum_{k=1}^{N} \lambda_k \sigma_k(x) \geq 0$$

relaxation = approximation, λ_i = Lagrange coefficients

Lagrangian relaxation, equality constraints

$$\forall x \in \mathbb{V} : \left(\bigwedge_{k=1}^{N} \sigma_k(x) = 0 \right) \Rightarrow (\sigma_0(x) \geq 0)$$

$\Leftarrow$ soundness (Lagrange)

$$\exists \lambda \in [1, N] \mapsto \mathbb{R}^+ : \forall x \in \mathbb{V} : \sigma_0(x) - \sum_{k=1}^{N} \lambda_k \sigma_k(x) \geq 0$$

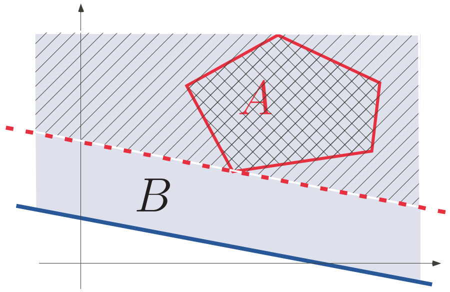
$$\wedge \ \exists \lambda' \in [1, N] \mapsto \mathbb{R}^+ : \forall x \in \mathbb{V} : \sigma_0(x) + \sum_{k=1}^{N} \lambda'_k \sigma_k(x) \geq 0$$

$$\Leftrightarrow (\lambda'' = \frac{\lambda' - \lambda}{2})$$

$$\exists \lambda'' \in [1, N] \mapsto \mathbb{R} : \forall x \in \mathbb{V} : \sigma_0(x) - \sum_{k=1}^{N} \lambda''_k \sigma_k(x) \geq 0$$

Example: affine Farkas' lemma, informally

- An application of Lagrangian relaxation to the case when A is a polyhedron



Example: affine Farkas' lemma, formally

- Formally, if the system $Ax + b \geq 0$ is feasible then

$$\forall x : Ax + b \geq 0 \Rightarrow cx + d \geq 0$$

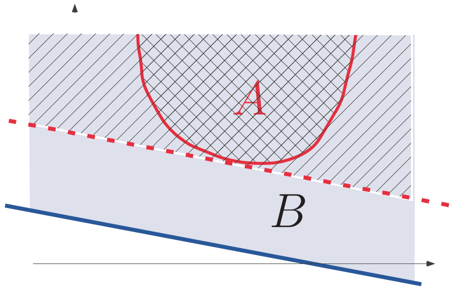
$\Leftarrow$ (soundness, Lagrange)

$\Rightarrow$ (completeness, Farkas)

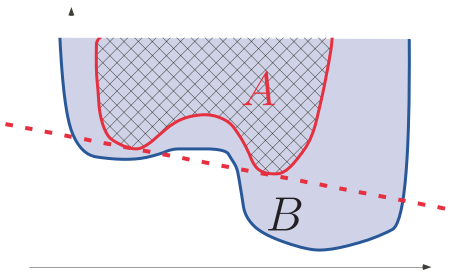
$$\exists \lambda \geq 0 : \forall x : cx + d - \lambda(Ax + b) \geq 0 \ .$$

Yakubovich's S-procedure, informally

– An application of Lagrangian relaxation to the case when A is a quadratic form



Incompleteness (convex case)



Yakubovich's S-procedure, completeness cases

– The constraint $\sigma(x) \geq 0$ is *regular* if and only if $\exists \xi \in \mathbb{V} : \sigma(\xi) > 0$.

– The S-procedure is lossless in the case of one regular quadratic constraint:

$$\forall x \in \mathbb{R}^n : x^\top P_1 x + 2q_1^\top x + r_1 \geq 0 \Rightarrow x^\top P_0 x + 2q_0^\top x + r_0 \geq 0$$

$\Leftarrow$ (Lagrange)

$\Rightarrow$ (Yakubovich)

$$\exists \lambda \geq 0 : \forall x \in \mathbb{R}^n : x^\top \left(\begin{bmatrix} P_0 & q_0 \\ q_0^\top & r_0 \end{bmatrix} - \lambda \begin{bmatrix} P_1 & q_1 \\ q_1^\top & r_1 \end{bmatrix} \right) x \geq 0.$$

Floyd's method for termination of `while B do C`

Find an $\mathbb{R}/\mathbb{Q}/\mathbb{Z}$-valued unkown rank function r which is:

– *Nonnegative*: $\exists \lambda \in [1, N] \mapsto \mathbb{R}^{+i}$:

$$\forall\ x_0, x : r(x_0) - \sum_{i=1}^{N} \lambda_i \sigma_i(x_0, x) \geq 0$$

– *Strictly decreasing*: $\exists \eta > 0 : \exists \lambda' \in [1, N] \mapsto \mathbb{R}^{+i}$:

$$\forall\ x_0, x : (r(x_0) - r(x) - \eta) - \sum_{i=1}^{N} \lambda'_i \sigma_i(x_0, x) \geq 0$$

Idea 4

Parametric abstraction of the ranking function r

Parametric abstraction

– How can we compute the ranking function r?

$\rightarrow$ parametric abstraction:

1. Fix the form r_a of the function r a priori, in term of unkown parameters a
2. Compute the parameters a numerically

– Examples:

$r_a(x) = a.x^\top$	linear
$r_a(x) = a.(x\ 1)^\top$	affine
$r_a(x) = (x\ 1).a.(x\ 1)^\top$	quadratic

Floyd's method for termination of `while B do C`

Find $\mathbb{R}/\mathbb{Q}/\mathbb{Z}$-valued unkown parameters a, such that:

– *Nonnegative*: $\exists \lambda \in [1, N] \mapsto \mathbb{R}^{+i}$:

$$\forall\ x_0, x : r_a(x_0) - \sum_{i=1}^{N} \lambda_i \sigma_i(x_0, x) \geq 0$$

– *Strictly decreasing*: $\exists \eta > 0 : \exists \lambda' \in [1, N] \mapsto \mathbb{R}^{+i}$:

$$\forall\ x_0, x : (r_a(x_0) - r_a(x) - \eta) - \sum_{i=1}^{N} \lambda'_i \sigma_i(x_0, x) \geq 0$$

Idea 5

Eliminate the universal quantification $\forall$ using linear matrix inequalities (LMIs)

Mathematical programming

$$\exists x \in \mathbb{R}^n: \quad \bigwedge_{i=1}^{N} g_i(x) \geqslant 0$$

$$[\text{Minimizing} \quad f(x)]$$

feasibility problem : find a solution to the constraints

optimization problem : find a solution, minimizing $f(x)$

Example: Linear programming

$$\exists x \in \mathbb{R}^n: \quad Ax \geqslant b$$

$$[\text{Minimizing} \quad cx]$$

Feasibility

– feasibility problem: find a solution $s \in \mathbb{R}^n$ to the optimization program, such that $\bigwedge_{i=1}^{N} g_i(s) \geq 0$, or to determine that the problem is *infeasible*
– feasible set: $\{x \mid \bigwedge_{i=1}^{N} g_i(x) \geq 0\}$
– a feasibility problem can be converted into the optimization program

$$\min\{-y \in \mathbb{R} \mid \bigwedge_{i=1}^{N} g_i(x) - y \geq 0\}$$

Semidefinite programming

$$\exists x \in \mathbb{R}^n: \quad M(x) \succcurlyeq 0$$

$$[\text{Minimizing} \quad cx]$$

Where the linear matrix inequality (LMI) is

$$M(x) = M_0 + \sum_{k=1}^{n} x_k M_k$$

with symetric matrices ($M_k = {M_k}^\top$) and the positive semidefiniteness is

$$M(x) \succcurlyeq 0 = \forall X \in \mathbb{R}^N : X^\top M(x) X \geq 0$$

Semidefinite programming, once again

Feasibility is:

$$\exists x \in \mathbb{R}^n: \forall X \in \mathbb{R}^N : X^\top \left(M_0 + \sum_{k=1}^{n} x_k M_k \right) X \geq 0$$

of the form of the formulæ we are interested in for programs which semantics can be expressed as *LMIs*:

$$\bigwedge_{i=1}^{N} \sigma_i(x_0, x) \geqslant_i 0 = \bigwedge_{i=1}^{N} (x_0 \; x \; 1) M_i (x_0 \; x \; 1)^\top \geqslant_i 0$$

Floyd's method for termination of `while B do C`

Find $\mathbb{R}/\mathbb{Q}/\mathbb{Z}$-valued unkown parameters a, such that:

– *Nonnegative*: $\exists \lambda \in [1, N] \mapsto \mathbb{R}^{+i}$:

$$\forall\, x_0, x : r_a(x_0) - \sum_{i=1}^{N} \lambda_i (x_0\ x\ 1) M_i (x_0\ x\ 1)^\top \geq 0$$

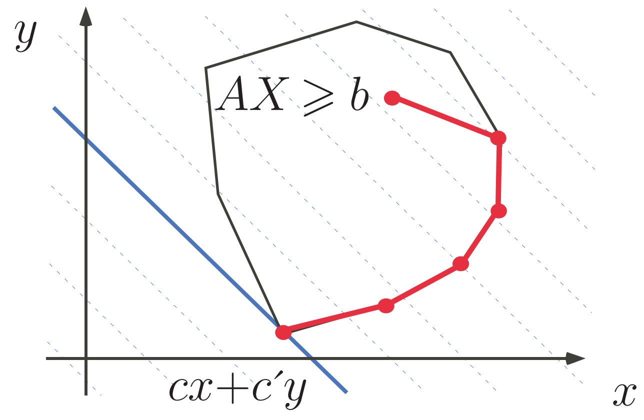
– *Strictly decreasing*: $\exists \eta > 0 : \exists \lambda' \in [1, N] \mapsto \mathbb{R}^{+i}$:

$$\forall\, x_0, x : (r_a(x_0) - r_a(x) - \eta) - \sum_{i=1}^{N} \lambda'_i (x_0\ x\ 1) M_i (x_0\ x\ 1)^\top \geq 0$$

Idea 6

Solve the convex constraints by semidefinite programming

The simplex for linear programming



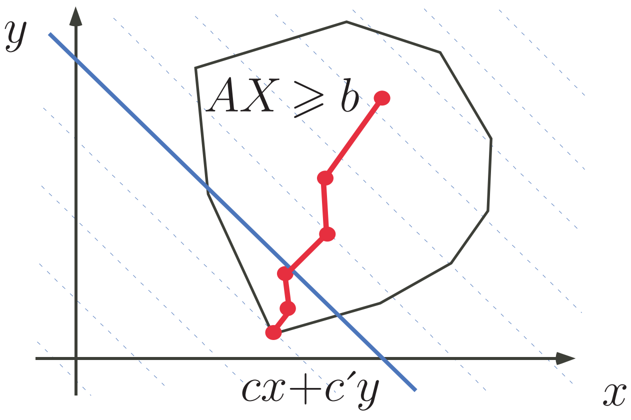
Dantzig 1948, exponential in worst case, good in practice

Polynomial methods

Ellipsoid method : Khachian 1979, polynomial in worst case but not good in practice

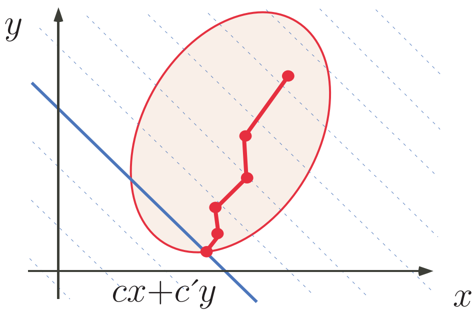
Interior point method : Kamarkar 1984, polynomial in worst case and good in practice (hundreds of thousands of variables)

The interior point method



Interior point method for semidefinite programming

– Nesterov & Nemirovskii 1988, polynomial in worst case and good in practice (thousands of variables)



– Various path strategies e.g. “stay in the middle”

Semidefinite programming solvers

Numerous solvers available under MATHLAB®, a.o.:

- `lmilab`: P. Gahinet, A. Nemirovskii, A.J. Laub, M. Chilali
- `Sdplr`: S. Burer, R. Monteiro, C. Choi
- `Sdpt3`: R. Tütüncü, K. Toh, M. Todd
- `SeDuMi`: J. Sturm
- `bnb`: J. Löfberg (integer semidefinite programming)

Common interfaces to these solvers, a.o.:

- `Yalmip`: J. Löfberg

Sometime need some help (feasibility radius, shift,...)

Linear program: termination of Euclidean division

```
» clear all
% linear inequalities
%       y0 q0 r0
Ai =  [ 0  0  0; 0  0  0;
        0  0  0];
%        y  q  r
Ai_ = [ 1  0  0;  % y - 1 >= 0
        0  1  0;  % q - 1 >= 0
        0  0  1]; % r >= 0
bi = [-1; -1; 0];
% linear equalities
%       y0 q0 r0
Ae =  [ 0 -1  0;  % -q0 + q -1 = 0
       -1  0  0;  % -y0 + y = 0
        0  0 -1]; % -r0 + y + r = 0
%        y  q  r
Ae_ = [ 0  1  0; 1  0  0;
        1  0  1];
be = [-1; 0; 0];
```

Iterated forward/backward polyhedral analysis:

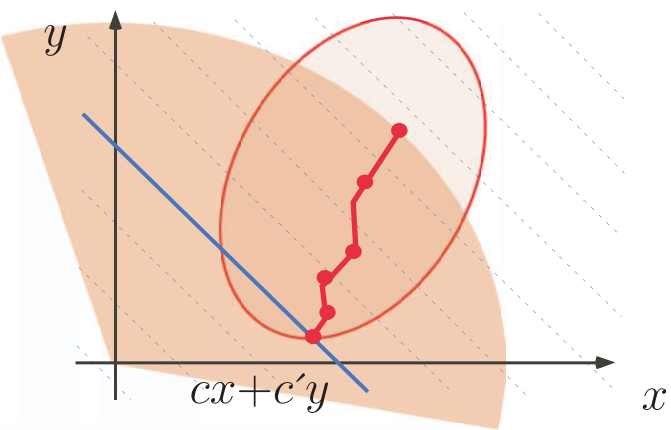
```
{y>=1}
q := 0;
{q=0,y>=1}
r := x;
{x=r,q=0,y>=1}
while (y <= r) do
   {y<=r,q>=0}
   r := r - y;
   {r>=0,q>=0}
   q := q + 1
   {r>=0,q>=1}
od
{q>=0,y>=r+1}
```

```
» [N Mk(:,:,:)]=linToMk(Ai, Ai_, bi);
» [M Mk(:,:,N+1:N+M)]=linToMk(Ae, Ae_, be);
» [v0,v]=variables('y','q','r');
» display_Mk(Mk, N, v0, v);
 +1.y -1 >= 0
 +1.q -1 >= 0
 +1.r >= 0
 -1.q0 +1.q -1 = 0
 -1.y0 +1.y = 0
 -1.r0 +1.y +1.r = 0
» [diagnostic,R] = termination(v0, v, Mk, N, 'integer', 'quadratic');
» disp(diagnostic)
   termination (bnb)
» intrank(R, v)
```

```
r(y,q,r) = -2.y +2.q +6.r
```

Floyd's proposal $r(x, y, q, r) = x - q$ is more intuitive but requires to discover the nonlinear loop invariant $x = r + qy$.

Imposing a feasibility radius



Quadratic program: termination of factorial

Program:

```
n := 0;
f := 1;
while (f <= N) do
    n := n + 1;
    f := n * f
od
```

LMI semantics:

```
-1.f0 +1.N0 >= 0
+1.n0 >= 0
+1.f0 -1 >= 0
-1.n0 +1.n -1 = 0
+1.N0 -1.N = 0
-1.f0.n +1.f = 0
```

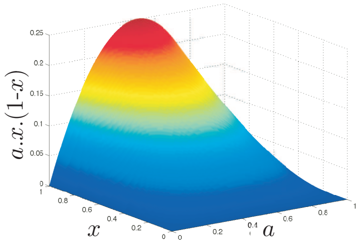
```
r(n,f,N) = -9.993455e-01.n +4.346533e-04.f
           +2.689218e+02.N +8.744670e+02
```


Idea 7

Convex abstraction of non-convex constraints

Semidefinite programming relaxation for polynomial programs

```
eps = 1.0e-9;
while (0 <= a) & (a <= 1 - eps)
        & (eps <= x) & (x <= 1) do
   x := a*x*(1-x)
od
```



Write the verification conditions in polynomial form, use SOS solver to relax in semidefinite programming form. SOStool+SeDuMi:

```
r(x) = 1.222356e-13.x + 1.406392e+00
```


Considering More General Forms of Programs

Handling disjunctive loop tests and tests in loop body

– By case analysis
– and "conditional Lagrangian relaxation" (Lagrangian relaxation in each of the cases)

Loop body with tests

```
while (x < y) do
  if (i >= 0) then
     x := x+i+1
  else
     y := y+i
  fi
od
```

$\longrightarrow$ case analysis: $\begin{cases} \mathtt{i} \geq 0 \\ \mathtt{i} < 0 \end{cases}$

lmilab:

```
r(i,x,y) = -2.252791e-09.i -4.355697e+07.x +4.355697e+07.y
                                            +5.502903e+08
```

Quadratic termination of linear loop

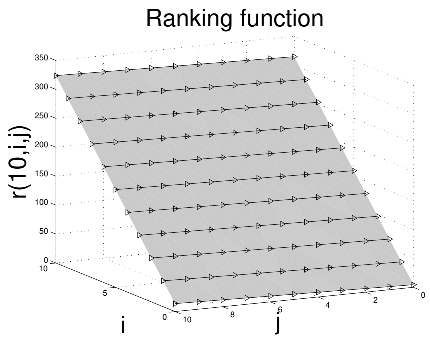
```
{n>=0}
i := n; j := n;
while (i <> 0) do
  if (j > 0) then
    j := j - 1
  else
    j := n; i := i - 1
  fi
od
```

⟵ termination precondition determined by iterated forward/backward polyhedral analysis

sdplr (with feasibility radius of 1.0e+3):

```
r(n,i,j) = +7.024176e-04.n^2 +4.394909e-05.n.i ...
           -2.809222e-03.n.j +1.533829e-02.n ...
           +1.569773e-03.i^2 +7.077127e-05.i.j   ...
           +3.093629e+01.i -7.021870e-04.j^2 ...
           +9.940151e-01.j +4.237694e+00
```

Successive values of r(n, i, j) for n = 10 on loop entry



Handling nested loops

- by induction on the loop depth
- use an iterated forward/backward symbolic analysis to get a necessary termination precondition
- use a forward symbolic symbolic analysis to get the semantics of a loop body
- use Lagrangian relaxation and semidefinite programming to get the ranking function

Example of termination of nested loops: Bubblesort inner loop

```
...
+1.i' -1 >= 0
+1.j' -1 >= 0
+1.n0' -1.i' >= 0
-1.j +1.j' -1 = 0
-1.i +1.i' = 0
-1.n +1.n0' = 0
+1.n0 -1.n0' = 0
+1.n0' -1.n' = 0
...
```

Iterated forward/backward polyhedral analysis followed by forward analysis of the body:

```
assume (n0 = n & j >= 0 & i >= 1 & n0 >= i & j <> i);
{n0=n,i>=1,j>=0,n0>=i}
assume (n01 = n0 & n1 = n & i1 = i & j1 = j);
{j=j1,i=i1,n0=n1,n0=n01,n0=n,i>=1,j>=0,n0>=i}
j := j + 1
{j=j1+1,i=i1,n0=n1,n0=n01,n0=n,i>=1,j>=1,n0>=i}
```

```
termination (lmilab)
r(n0,n,i,j) = +434297566.n0 +226687644.n -72551842.i
                                          -2.j +2147483647
```

Example of termination of nested loops: Bubblesort outer loop

```
...
+1.i' +1 >= 0
+1.n0' -1.i' -1 >= 0
+1.i' -1.j' +1 = 0
-1.i +1.i' +1 = 0
-1.n +1.n0' = 0
+1.n0 -1.n0' = 0
+1.n0' -1.n' = 0
...
```

Iterated forward/backward polyhedral analysis followed by forward analysis of the body:

```
  assume (n0=n & i>=0 & n>=i & i <> 0);
{n0=n,i>=0,n0>=i}
  assume (n01=n0 & n1=n & i1=i & j1=j);
{j1=j,i=i1,n0=n1,n0=n01,n0=n,i>=0,n0>=i}
  j := 0;
  while (j <> i) do
      j := j + 1
  od;
  i := i - 1
{i+1=j,i+1=i1,n0=n1,n0=n01,n0=n,i+1>=0,n0>=i+1}
termination (lmilab)
r(n0,n,i,j) = +24348786.n0 +16834142.n +100314562.i +65646865
```


Handling nondeterminacy

– By case analysis

– Same for concurrency by interleaving

– Same with fairness by nondeterministic interleaving with encoding of an explicit scheduler

Termination of a concurrent program

```
[|  1: while [x+2 < y] do
    2:     [x := x + 1]
       od
    3:
||
    1: while [x+2 < y] do
    2:     [y := y - 1]
       od
    3:
|]
```

interleaving $\longrightarrow$

```
while (x+2 < y) do
  if ?=0 then
    x := x + 1
  else if ?=0 then
    y := y - 1
  else
    x := x + 1;
    y := y - 1
  fi fi
od
```

penbmi: `r(x,y) = 2.537395e+00.x+-2.537395e+00.y+ -2.046610e-01`

Termination of a fair parallel program

```
[[ while [(x>0)|(y>0) do x := x - 1] od ||
   while [(x>0)|(y>0) do y := y - 1] od ]]
```

interleaving + scheduler $\longrightarrow$

`{m>=1}` ← termination precondition determined by iterated forward/backward polyhedral analysis

```
t := ?;
assume (0 <= t & t <= 1);
s := ?;
assume ((1 <= s) & (s <= m));
while ((x > 0) | (y > 0)) do
  if (t = 1) then
     x := x - 1
  else
     y := y - 1
  fi;
  s := s - 1;
  if (s = 0) then
    if (t = 1) then
       t := 0
    else
       t := 1
    fi;
    s := ?;
    assume ((1 <= s) & (s <= m))
  else
    skip
  fi
od;;
```

penbmi: `r(x,y,m,s,t) = +1.000468e+00.x +1.000611e+00.y +2.855769e-02.m -3.929197e-07.s +6.588027e-06.t +9.998392e+03`

Relaxed Parametric Invariance Proof Method

Abstraction

- Express loop semantics as a conjunction of LMI constraints (by relaxation for polynomial semantics)
- Eliminate the conjunction and implication by Lagrangian relaxation
- Fix the form of the unkown invariant by parametric abstraction

... we get ...

Floyd's method for invariance

Given a loop precondition P, find an unkown loop invariant I such that:

- The invariant is *initial*:

$$\forall\ x : P(x) \Rightarrow \underset{\uparrow}{I}\,(x)$$

- The invariant is *inductive*:

$$\forall\ x, x' :\ \underset{\substack{\uparrow\\ ???}}{I}\,(x) \wedge [\![\mathtt{B};\mathtt{C}]\!](x, x') \Rightarrow \underset{\uparrow}{I}\,(x')$$

Floyd's method for numerical programs

Find $\mathbb{R}/\mathbb{Q}/\mathbb{Z}$-valued unkown parameters a, such that:

- The invariant is *initial*: $\exists \mu \in \mathbb{R}^+$:

$$\forall\ x : I_a(x) - \mu.P(x) \geq 0$$

- The invariant is *inductive*: $\exists \lambda \in [0, N] \longrightarrow \mathbb{R}^+$:

$$\forall\ x, x' : I_a(x') - \underset{\uparrow}{\lambda_0}.\underset{\uparrow}{I_a}(x) - \sum_{k=1}^{N} \lambda_k.\sigma_k(x, x') \geq 0$$

bilinear in λ_0 and a

Idea 8

Solve the bilinear matrix inequality (BMI) by semidefinite programming

Bilinear matrix inequality (BMI) solvers

$$\exists x \in \mathbb{R}^n : \bigwedge_{i=1}^{m} \left(M_0^i + \sum_{k=1}^{n} x_k M_k^i + \sum_{k=1}^{n} \sum_{\ell=1}^{n} x_k x_\ell N_{k\ell}^i \succcurlyeq 0 \right)$$

[Minimizing $x^\top Q x + cx$]

Two solvers available under MATHLAB®:

- PenBMI: M. Kočvara, M. Stingl
- bmibnb: J. Löfberg

Common interfaces to these solvers:

- Yalmip: J. Löfberg

Example: linear invariant

Program:

```
i := 2; j := 0;
while (??) do
  if (??) then
    i := i + 4
  else
    i := i + 2;
    j := j + 1
  fi
od;
```

- Invariant:

  `+2.14678e-12*i -3.12793e-10*j +0.486712 >= 0`

- Less natural than $i - 2j - 2 \geq 0$
- Alternative:
  - Determine parameters (a) by other methods (e.g. random interpretation)
  - Use BMI solvers to *check* for invariance

Conclusion

Constraint resolution failure

– infeasibility of the constraints does not mean “non termination” or “non invariance” but simply failure

– inherent to abstraction!

Numerical errors

– LMI/BMI solvers do numerical computations with rounding errors, shifts, etc

– ranking function is subject to numerical errors

– the hard point is to discover a candidate for the ranking function

– much less difficult, when the ranking function is known, to re-check for satisfaction (e.g. by static analysis)

– not very satisfactory for invariance (checking only ???)

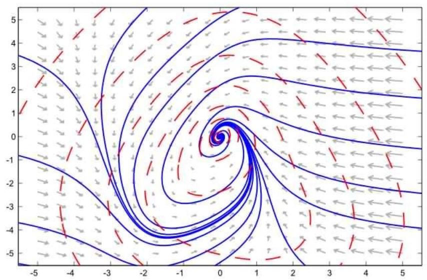
Related work

– Linear case (Farkas lemma):

- Invariants: Sankaranarayanan, Spima, Manna (CAV’03, SAS’04, heuristic solver)
- Termination: Podelski & Rybalchenko (VMCAI’03, Lagrange coefficients eliminated by hand to reduce to linear programming so no disjunctions, no tests, etc)
- Parallelization & scheduling: Feautrier, easily generalizable to nonlinear case

Seminal work

– LMI case, Lyapunov 1890, *“an invariant set of a differential equation is stable in the sense that it attracts all solutions if one can find a function that is bounded from below and decreases along all solutions outside the invariant set”*.



THE END, THANK YOU