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ASYNCHRONOUS ITERATIVE METHODS FOR SOLVING A FIXED POINT SYSTEM OF MONOTONE EQUATIONS IN A COMPLETE LATTICE

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ASYNCHRONOUS ITERATIVE METHODS FOR SOLVING A FIXED POINT SYSTEM OF MONOTONE EQUATIONS IN A COMPLETE LATTICE

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Abstract. It is shown that the class of asynchronous iterative methods and asynchronous iterative methods with memory can be used to solve a fixed point system of monotone equations in a complete lattice. The rather technical proofs use no additional hypotheses (such as continuity or chain conditions). These iterative methods correspond to a parallel algorithm for solving the system of equations on a multiprocessor system with no synchronization between cooperating processes.

Key Words and Phrases : fixed point, system of monotone equations in a complete lattice, asynchronous iterative methods, asynchronous iterative methods with memory.

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1. INTRODUCTION AND NOTATIONS

Let $L(\sqsubseteq,\bot,\top,\sqcup,\sqcap)$ be a non-empty *complete lattice* with *partial ordering* $\sqsubseteq$, *least upper bound* $\sqcup$, *greatest lower bound* $\sqcap$. The *infimum* $\bot$ of L is $\sqcap L$, the *supremum* $\top$ of L is $\sqcup L$. Let F be a *monotone* operator from $L^n(\sqsubseteq,\bot,\top,\sqcup,\sqcap)$ into itself (i.e. $\forall X,Y\in L^n$, $\{X\sqsubseteq Y\} \Rightarrow \{F(X)\sqsubseteq F(Y)\}$). Tarski[7]'s theorem states that the set of *fixed points* of F (solutions to the equation $X=F(X)$) is a non-empty complete lattice with ordering $\sqsubseteq$.

Let μ be the smallest ordinal such that the class $\{\delta:\delta\in\mu\}$ has a cardinality greater than the cardinality of L^n. Cousot[3] defines constructively the fixed points of F by means of the following μ-termed transfinite sequences :

The *iteration sequence for* F *starting with* $D \in L^n$ is the μ-termed sequence $<B^\delta, \delta\in\mu>$ of elements of L^n defined by transfinite recursion in the following way :

- $B^0 = D$
- $B^\delta = F(B^{\delta-1})$ for every successor ordinal $\delta\in\mu$
- $B^\delta = \bigsqcup_{\alpha<\delta} B^\alpha$ for every limit ordinal $\delta\in\mu$

We say that the sequence is *stationary* iff $\{\exists\varepsilon\in\mu : \{\forall\beta\in\mu, \{\beta\geq\varepsilon\}\Rightarrow\{B^\varepsilon=B^\beta\}\}\}$ in which case the *limit* of the sequence denoted by *lis*$(F)(D)$ is defined to be B^ε.

A sufficient condition for the iteration sequence for F starting with D to be stationary is that D is a prefixed point of F ($D\sqsubseteq F(D)$). In this case $<B^\delta,\delta\in\mu>$ is an increasing chain, its limit *lis*$(F)(D)$ is the least of the fixed points of F greater than D. It is also greater than any fixed point of F less than D (if such fixed points exist), (Cousot[3]). In practice we are often interested by the least fixed point of F which is *lis*$(F)(\bot)$.

We now consider the case when the fixed point equation $X=F(X)$ is of the form :

$$\begin{cases} X_i = F_i(X) = F_i(X_1,\ldots,X_n) \\ i = 1..n \end{cases}$$

where each F_i, $i=1..n$ is a fixed monotone function of L^n into L.

(If $X \in L^n$ then X_i denotes the i-th component of X. If $X \in (L^n)^m$ then $(X_j)_i$ denotes the i-th component of the j-th component of X. If $\langle X^\delta, \delta \in \mu \rangle$ is a μ-termed sequence, X^δ denotes its δ-th term. Therefore X_i^δ is the i-th component of the δ-th term of the sequence $\langle X^\delta, \delta \in \mu \rangle$).

The iteration sequence for F starting with D is then defined by :

- $X^0 = D$
- $X_i^\delta = F_i(X^{\delta-1})$ for every $i \in \{1,\dots,n\}$ and every successor ordinal $\delta \in \mu$
- $X^\delta = \bigsqcup_{\alpha<\beta} X^\alpha$ for every limit ordinal $\delta \in \mu$.

This is nothing else than Jacobi's method of successive approximations (where limit ordinals are also considered).

More generally, a *chaotic iteration sequence for* F *starting with* $D \in L^n$ *and defined by* the μ-termed sequence $\langle J^\delta, \delta \in \mu \rangle$ of subsets of $\{1,\dots,n\}$ with maximal residue (that is $\{\forall \delta \in \mu,\ \forall i \in \{1,\dots,n\},\ \exists \alpha \in \mu : (\alpha \geq \delta)$ and $(i \in J^\alpha)\}$) is the μ-termed sequence $\langle X^\delta, \delta \in \mu \rangle$ defined by transfinite recursion as follows :

- $X^0 = D$
- $X_i^\delta = X_i^{\delta-1}$ for every successor ordinal $\delta \in \mu$ and every $i \in (\{1,\dots,n\} - J^\delta)$
- $X_i^\delta = F_i(X^{\delta-1})$ for every successor ordinal $\delta \in \mu$ and every $i \in J^\delta$
- $X^\delta = \bigsqcup_{\alpha<\delta} X^\alpha$ for every limit ordinal $\delta \in \mu$

(In numerical analysis a similar definition is given in Robert[6] for finite sequences).

Jacobi's iteration method consists in choosing $\{\forall \delta \in \mu,\ J^\delta = \{1,\dots,n\}\}$ whereas Gauss-Seidel's iteration method is equivalent to $J^\delta = \{1\}$ if $\delta = 1$ or δ is successor of a limit ordinal and $J^\delta = \{1 + (j \ \underline{\text{modulo}}\ n)\}$ if δ is a successor ordinal and $J^{\delta-1} = \{j\}$.

It is known that without sufficient hypothesis on L^n and F all chaotic iteration methods are not equivalent. For example one of the Jacobi's and Gauss-Seidel's sequences may be stationary whereas the other is not, (Robert[5]). However when F is monotone and L is a complete lattice Cousot[2] proves that any chaotic iteration sequence for F starting with a prefixed point D of F is stationary its limit beeing *lis*(F)(D). Yet this proof was based on the additional assumptions that F is *continuous* (for any increasing chain $<C^i, i \in E>$ we have $F(\bigsqcup_{i \in E} C^i) = \bigsqcup_{i \in E} F(C^i)$) and that the length of the periods of $<J^\delta, \delta \in \mu>$ was uniformly bounded $\{\exists m \in \omega : \{\forall \delta \in \mu, \forall i \in \{1,\ldots,r\}, \exists \alpha \in \mu : (\delta \leq \alpha \leq \delta+m)$ $\underline{\text{and}}$ $(i \in J^\alpha)\}\}$. Also in the model of chaotic iterations all components X_i^δ $(i \in J^\delta)$ are evaluated in term of the previous iterate $X^{\delta-1}$ and this must be done by a single computation process or by several synchronized parallel processes.

The purpose of this paper is to eliminate the previous restrictions and mainly to account for the parallel implementation of iterative methods on a multiprocessor computer system without synchronization between cooperating processes.

An important domain of possible application of the results contained in this paper is the one of global program analysis and optimization techniques. Most often these compiling techniques consist in solving a fixed point system of monotone equations in a complete lattice. Showing that such systems of equations can be solved by asynchronous iterative methods implies that all classical program analysis methods which are based on sequential iterative algorithms are amenable to a parallel implementation on a multiprocessor system without synchronization between cooperating processes. This enables the compiler writters to follow the evolution of the hardware technology without major revision of the known techniques.

2. ASYNCHRONOUS ITERATIONS

2.1 DEFINITION

- Let $<J^\delta, \delta \in Ord>$ be an Ord-termed sequence of subsets of $\{1,\ldots,n\}$ such that :

  (a) $\{\forall \delta \in Ord, \forall i \in \{1,\ldots,n\}, \exists \alpha \geq \delta : i \in J^\alpha\}$

- Let $<S^\delta, \delta \in Ord>$ be an Ord-termed sequence of elements of Ord^n such that :

  (b) $\{\forall i \in \{1,\ldots,n\}, \forall \delta \in Ord, S_i^\delta < \delta\}$

(c) $\{\forall \delta \in Ord, \forall i \in \{1,\ldots,n\}, \exists \beta \geq \delta : \{\forall \alpha \geq \beta, \delta \leq S_i^\alpha\}\}$

(d) $\{\forall \beta, \delta \in Ord, \{\beta \text{ is a limit ordinal and } \beta < \delta\} \Rightarrow \{\forall i \in \{1,\ldots,n\}, \beta \leq S_i^\delta\}\}$

- Let F be a monotone operator of the complete lattice L^n into itself. *An asynchronous iteration sequence for* F *starting with* $D \in L^n$ *and defined by* $<J^\delta, \delta \in Ord>$ and $<S^\delta, \delta \in Ord>$ is the Ord-termed sequence $<X^\delta, \delta \in Ord>$ defined by transfinite recursion as follows :
  - $X^0 = D$
  - $X_i^\delta = X_i^{\delta-1}$ for every successor ordinal δ and every $i \in (\{1,\ldots,n\} - J^\delta)$
  - $X_i^\delta = F_i(X_1^{S_1^\delta}, \ldots, X_n^{S_n^\delta})$ for every successor ordinal δ and every $i \in J^\delta$
  - $X^\delta = \bigsqcup_{\alpha<\delta} X^\alpha$ for every limit ordinal δ.

(In numerical analysis, a similar definition is given in Baudet[1] for finite sequences).
For example, the choice $S_i^\delta = \delta-1$ for every successor ordinal δ and every index $i=1,\ldots,n$ corresponds to the definition of a chaotic iteration sequence given in paragraph 1.

2.2 INTUITIVE EXPLANATION OF THE DEFINITION

Intuitively, the above definition accounts for computations on an asynchronous multiprocessor. A global memory X initialized with D is available. Each processor can read or write any component X_i of the global memory X. These operations are indivisible that is the reading and writing operations of a component of X are mutually exclusive in time. Therefore the reading and writing of any X_i can be considered as instantaneous operations.

Interpret $<\delta, \delta \in Ord>$ as an increasing sequence of time instants where reading or writing of some component of X take place. When idle a processor is assigned to the evaluation of any component of the system of equations. Then definition 2.1 states that at time δ a certain number of processors terminate the evaluation of all components i of the system of equations for which $i \in J^\delta$. Therefore the corresponding value X_i^δ is instantaneously written in the memory X_i. This evaluation of X_i^δ consisted in reading the value $X_1^{S_1^\delta}$ of the memory X_1 at time $S_1^\delta, \ldots,$ in reading X_n at time S_n^δ, in applying the operator F_i to $X_1^{S_1^\delta}, \ldots, X_n^{S_n^\delta}$ and writing the value $X_{i_\delta}^\delta = F_i(X_1^{S_1^\delta}, \ldots, X_n^{S_n^\delta})$ at time δ in X_i. The components X_j of X for which $j \notin J^\delta$ are not modified at time δ.

Notice that the reading of every value $X_j^{S_j^\delta}$ used in the computation of X_i^δ necessarily takes place before the value $X_i^\delta = F_i(X_1^{S_1^\delta}, \ldots, X_n^{S_n^\delta})$ is written in X_i. This justifies condition 2.1.(b).

According to definition 2.1 no synchronization between the cooperating processes is needed, and the scheduling policy (specifying which processors evaluate which components) is left opened. However the scheduling policy must be fair so that condition 2.1.(a) imposes that no component can be abandoned forever.

We also assume that the evaluation of every $F_i(X_1^{S_1^\delta}, \ldots, X_n^{S_n^\delta})$ takes a finite (but not necessarily bounded) period of time and that if a processor collapses the scheduling policy will be modified in order to have its task performed by the remaining processors. Therefore for every δ, $\max\limits_{i=1}^{n}(\delta - S_i^\delta)$ must be bound (but not necessarily uniformly). Consequently condition 2.1.(c) states that for any δ there exists β such that after time β no processor can terminate a computation that began at time δ.

In practive we should also require that the sequence $<X^\delta, \delta \in Ord>$ is stationary after a finite time ε. It is immediat from what follows that for example this would be the case when L satisfies the ascending chain condition. Since the possible hypotheses guaranteeing the termination of the computations are not unique it is better from a mathematical point of view to ignore this practical limitation and consider limit ordinals.

The consideration of an infinite computation sequence does not eliminate our plausible hypothesis that the elementary computations (that is the evaluation of a component) must last a finite time. Hence we state that condition 2.1.(d) is necessary to take account of this fact. By reductio ad absurdum assume that the computation of X_i^δ beginning at time $\alpha=\underline{\min_{i=1}^{n}}(S_i^\delta)$ takes r units of time (where r is integer since the duration of this computation can be arbitrarily long but finite). We have $\alpha+r=\delta$. Assume that β is a non zero limit ordinal and there is j such that $S_j^\delta<\beta<\delta$, then $\alpha<\beta<\delta$. Since β is a limit number $\alpha+1<\beta$ so that by finite induction $\alpha+r<\beta<\delta$ in contradiction with $\alpha+r=\delta$.

2.3 CONVERGENCE THEOREM

THEOREM 2.3.1 An asynchronous iteration sequence corresponding to the monotone operator F of the complete lattice L^n into itself and starting with a prefixed point D of F is stationary, its limit is *lis*(F)(D).

Proof : The proof is similar to the one given in paragraph 3 for theorem 3.3.10 when m=1 in definition 3.1. *End of Proof.*

3. ASYNCHRONOUS ITERATIONS WITH MEMORY

3.1 DEFINITION

- Let $\langle J^\delta, \delta\in\text{Ord}\rangle$ be a transfinite sequence of subsets of $\{1,\dots,n\}$ such that :

  (a) $\{\forall\delta\in\text{Ord},\ \forall i\in\{1,\dots,n\},\ \exists\alpha\geq\delta\ :\ i\in J^\alpha\}$

- Let $\langle S^\delta, \delta\in\text{Ord}\rangle$ be a transfinite sequence of elements of $(\text{Ord}^n)^m$ such that :

  (b) $\{\forall i\in\{1,\dots,n\},\ \forall j\in\{1,\dots,m\},\ \forall\delta\in\text{Ord},\ (S_j^\delta)_i<\delta\}$

  (c) $\{\forall\delta\in\text{Ord},\ \forall i\in\{1,\dots,n\},\ \forall j\in\{1,\dots,m\},\ \exists\beta\geq\delta\ :\ \{\forall\alpha\geq\beta,\ \delta\leq(S_j^\alpha)_i\}\}$

  (d) $\{\forall\beta,\delta\in\text{Ord},\ \{\beta \text{ is a limit ordinal and } \beta<\delta\}\Rightarrow\{\forall i\in\{1,\dots,n\},\ \forall j\in\{1,\dots,m\},\ \beta\leq(S_j^\delta)_i\}$

- Let L^n be a complete lattice and F a monotone operator of $(L^n)^m$ into L^n. *An asynchronous iteration sequence with memory corresponding to* F, *starting with* D *and defined by* $\langle J^\delta,\delta\in\text{Ord}\rangle$ *and* $\langle S^\delta,\delta\in\text{Ord}\rangle$ is the Ord-termed sequence $\langle X^\delta,\delta\in\text{Ord}\rangle$ of elements of L^n defined by transfinite recursion as follows :

- $X^0 = D$
- $X_i^\delta = X_i^{\delta-1}$ for every successor ordinal δ and every $i\in(\{1,\ldots,n\}-J^\delta)$
- $X_i^\delta = F_i(Z^1,\ldots,Z^m)$ for every successor ordinal δ and every $i\in J^\delta$

  where

  $\{\forall j\in\{1,\ldots,m\},\ \forall i\in\{1,\ldots,n\},\ Z_i^j = X_i^{(S_j^\delta)_i}\}$
- $X^\delta = \bigsqcup\limits_{\alpha<\delta} X^\alpha$ for every limit ordinal δ

(In numerical analysis a similar definition is given in Baudet[1] for finite sequences). When m equals one definition 2.2 is equivalent to definition 2.1.

3.2 INTUITIVE EXPLANATION OF THE DEFINITION

Let σ be the function of L^n into $(L^n)^m$ such that $\{\forall X\in L^n, \forall i\in\{1,\ldots,m\}, (\sigma(X))_i=X\}$. Let F be a monotone operator of $(L^n)^m$ into L^n. We define a fixed point of F to be an element of L^n such that $X=F(X,\ldots,X)$ that is $X=F(\sigma(X))$. Since by definition σ is a monotone function of L^n into $(L^n)^m$ and by hypothesis F is a monotone function of $(L^n)^m$ into L^n the composition $F\circ\sigma$ is a monotone operator of the complete lattice L^n into itself. Hence by Tarski[7]'s theorem the set of fixed points of F which is equal to the set of fixed points of $F\circ\sigma$ is a non-empty complete lattice with the ordering $\sqsubseteq$ of L^n.

Whenever D is a prefixed point of $F\circ\sigma$ (that is $D\sqsubseteq F(D,\ldots,D)$) the iteration sequence for $F\circ\sigma$ is a stationary increasing chain, its limit *lis*($F\circ\sigma$)(D) is the least of the fixed points of F greater than D. *lis*($F\circ\sigma$)(D) is also greater than any fixed point of F less than D (if such fixed points exist), (Cousot[3]).

We will show that whenever $D\sqsubseteq F(\sigma(D))$ any asynchronous iteration sequence with memory corresponding to the function F and starting with D is stationary, its limit being *lis*($F\circ\sigma$)(D).

For a practical application of this result assume that we have to compute *lis*(f)(D) where $f\in L^n\rightarrow L^n$ and that we can find some m and

$F\epsilon(L^n)^m \to L^n$ such that $lis(f)(D)=lis(F\circ\sigma)(D)$. Then we can use any asynchronous iterative method with memory corresponding to F to compute $lis(f)(D)$.

For example suppose that $\{\forall X\epsilon L^n,\ f(X)=g(X)\sqcup h(X)\}$. Then $lis(f)(D)=lis(F\circ\sigma)(D)$ where $F(X,Y)=g(X)\sqcup h(Y)$. A possible iteration with memory for F is

$$\begin{cases} X^0 = \bot \\ X^1 = \bot \\ X^{\delta+2} = F(X^{\delta+1},X^{\delta}) \end{cases}$$

In turn this is equivalent to the two collateral iterations :

$$\begin{cases} X^0 = \bot \\ X^{\delta+1} = g(X^{\delta})\sqcup Y^{\delta} \end{cases} \qquad \begin{cases} Y^0 = \bot \\ Y^{\delta+1} = h(X^{\delta}) \end{cases}$$

which is a natural decomposition of the computations not describable by definition 2.1.

3.3 CONVERGENCE THEOREM

In the following we consider an asynchronous iteration sequence with memory $<X^{\delta},\delta\epsilon Ord>$ for $F\epsilon(L^n)^m \to L^n$ starting with a prefixed point D of F and defined by $<J^{\delta},\delta\epsilon Ord>$ and $<S^{\delta},\delta\epsilon Ord>$.

LEMMA 3.3.1. $\{\forall\delta\epsilon Ord,\ D\sqsubseteq X^{\delta}\}$

Proof :

- The lemma is obvious for $\delta=0$ since $\sqsubseteq$ is reflexive and $X^0=D$.
- Assume that for every $\beta<\delta$ we have $D\sqsubseteq X^{\beta}$.
  - if δ is a limit ordinal then we have $D=X^0\sqsubseteq \bigsqcup_{\alpha<\delta} X^{\alpha}=X^{\delta}$
  - otherwise, δ is a successor ordinal and then for every $i=1,\ldots,n$ we have :
    - if $i\notin J^{\delta}$ then $D_i\sqsubseteq X_i^{\delta-1}$ by induction hypothesis and $X_i^{\delta}=X_i^{\delta-1}$ by definition of an asynchronous iteration sequence with memory so that by transitivity we prove $D_i\sqsubseteq X_i^{\delta}$.
    - if $i\epsilon J^{\delta}$ then $D_i\sqsubseteq F_i(\sigma(D))$ since D is a prefixed point of F and $X_i^{\delta}=F_i(Z^1,\ldots,Z^m)$. $\forall j\epsilon\{1,\ldots,m\}$, $\forall i\epsilon\{1,\ldots,n\}$, $Z_i^j=X_i^{(S_j^{\delta})i}$ where by condition (b) $(S_j^{\delta})_i<\delta$ so that by induction

hypothesis $D_i \sqsubseteq Z_i^j$ and $\forall j \in \{1,\ldots,m\}$, $D \sqsubseteq Z^j$. By monotony of F_i this implies $F_i(\sigma(D)) \sqsubseteq F_i(Z^1,\ldots,Z^m)$ so that by transitivity we have $D_i \sqsubseteq X_i^\delta$.

By transfinite induction we conclude $\forall \delta \in \mathrm{Ord}$, $D \sqsubseteq X^\delta$.

End of Proof.

LEMMA 3.3.2. $\{\forall \delta \in \mathrm{Ord},\ X^\delta \sqsubseteq \mathit{lis}(F \circ \sigma)(D)\}$

Proof :

- Let us recall (Cousot[3]) that since $D \sqsubseteq F(\sigma(D))$, $\mathit{lis}(F \circ \sigma)(D)$ exists and is a fixed point of $F \circ \sigma$ greater than D. Hence the lemma is true for $\delta=0$ since $X^0=D$.
- Assume that the lemma is true for all $\beta<\delta$.
  - If δ is a limit ordinal then by induction hypothesis $\{\forall \beta<\delta,\ X^\beta \sqsubseteq \mathit{lis}(F \circ \sigma)(D)\}$ so that by definition of the least upper bound $\bigsqcup_{\beta<\delta} X^\beta \sqsubseteq \mathit{lis}(F \circ \sigma)(D)$. Since $X^\delta = \bigsqcup_{\beta<\delta} X^\beta$ we conclude by transitivity that $X^\delta \sqsubseteq \mathit{lis}(F \circ \sigma)(D)$.
  - Otherwise δ is a successor ordinal and for every $i=1,\ldots,n$ we have :
    - if $i \notin J^\delta$ then $X_i^\delta = X_i^{\delta-1} \sqsubseteq (\mathit{lis}(F \circ \sigma)(D))_i$
    - if $i \in J^\delta$ then $X_i^\delta = F_i(Z^1,\ldots,Z^m)$ where $\forall j \in \{1 \ldots m\}$, $\forall i \in \{1,\ldots,n\}$, $Z_i^j = X_i^{(S_j^\delta)_i}$ and $(S_j^\delta)_i < \delta$ so that $\forall j \in \{1,\ldots,m\}$, $Z^j \sqsubseteq \mathit{lis}(F \circ \sigma)(D)$ by induction hypothesis. By monotony we get $X_i^\delta = F_i(Z^1,\ldots,Z^m) \sqsubseteq F_i(\sigma(\mathit{lis}(F \circ \sigma)(D))) \sqsubseteq (\mathit{lis}(F \circ \sigma)(D))_i$.

By transfinite induction we conclude $\{\forall \delta \in \mathrm{Ord},\ X^\delta \sqsubseteq \mathit{lis}(F \circ \sigma)(D)\}$.

End of Proof.

DEFINITION 3.3.3. Condition 3.1.(a) implies that for every $\delta \in \mathrm{Ord}$ there is an ordinal $\Pi(\delta)$ defined by :

$$\Pi(\delta) = \underline{\min}\ \{\alpha:\ \delta \leq \alpha\ \underline{\text{and}}\ \{1,\ldots,n\} = \bigcup_{\beta=\delta}^{\alpha} J^\beta\}$$

(Intuitively between times δ and $\Pi(\delta)+1$, all components have been evaluated at least once, so that :

$\{\forall i \in \{1,\ldots,n\},\ \forall \delta \in \mathrm{Ord},\ \exists \beta :\ \delta \leq \beta \leq \Pi(\delta)\ \underline{\text{and}}\ i \in J^\beta\}$)

DEFINITION 3.3.4. Condition 3.1.(c) implies that for every $\delta\in\text{Ord}$, there is an ordinal $\lambda(\delta)$ defined by :

$$\lambda(\delta) = \underline{\min}\,\{\beta: \forall j\in\{1,\dots,m\},\ \forall i\in\{1,\dots,n\},\ \forall\alpha\geq\beta,\ \delta\leq(S_j^\alpha)_i\}$$

(Intuitively a computation which terminates at time $\lambda(\delta)+1$ cannot have read the necessary components before time δ so that :

$\{\forall j\in\{1,\dots,m\},\ \forall i\in\{1,\dots,n\},\ \forall\alpha\geq\lambda(\delta),\ \delta\leq(S_j^\alpha)_i\}$).

DEFINITION 3.3.5. $<\eta^\delta,\delta\in\text{Ord}>$ is the Ord-termed sequence of ordinals defined by transfinite recursion as follows :

- $\eta^0 = 0$
- $\eta^\delta = \Pi\lambda(\eta^{\delta-1})+1$ if δ is a successor ordinal
- $\eta^\delta = \bigcup_{\alpha<\delta}\eta^\alpha$ if δ is a limit ordinal

LEMMA 3.3.6. The sequence $<\eta^\delta,\delta\in\text{Ord}>$ is *normal* (that is *limiting* and *strictly increasing*).

Proof : The sequence is limiting (for every non-zero limit ordinal δ we have $\eta^\delta=\bigcup_{\alpha<\delta}\eta^\alpha$) and such that $\{\forall\delta,\eta^\delta<\eta^{\delta+1}\}$ therefore it is normal (Monk[4], theorem 12.6) hence strictly increasing $\{\forall\alpha,\ \forall\beta,\ (\alpha<\beta \Rightarrow \eta^\alpha<\eta^\beta)\}$. *End of Proof.*

LEMMA 3.3.7. For every limit ordinal δ, η^δ is also a limit ordinal.

Proof : $<\eta^\delta,\delta\in\text{Ord}>$ is normal and Monk[4], theorem 12.7. *End of Proof.*

DEFINITION 3.3.8. $<B^\delta,\delta\in\text{Ord}>$ is the Ord-termed sequence defined by transfinite recursion as follows :

- $B^0 = D$
- $B^\delta = F(\sigma(B^{\delta-1}))$ if δ is a successor ordinal
- $B^\delta = \bigsqcup_{\alpha<\delta} B^\alpha$ if δ is a limit ordinal

LEMMA 3.3.9. $\forall\beta,\delta\in\text{Ord},\ \{\beta\geq\eta^\delta\} \Rightarrow \{B^\delta\sqsubseteq X^\beta\}$.

Proof : The proof is by transfinite induction on δ.

Case 1. If $\delta=0$ then $\forall\beta\geq\eta^0=0$ we have by definition 3.3.8 and lemma 3.3.1 $B^0=D\sqsubseteq X^\beta$.

Case 2. Assume that δ is a successor ordinal and that the lemma 3.3.9 is true for every $\delta'<\delta$. We prove by transfinite induction on β that it is also true for δ.

Case 2.1. If $\beta=\eta^{\delta}$ where δ is a successor ordinal then by definition 3.3.5 $\beta=\Pi\lambda(\eta^{\delta-1})+1$. Then $\forall i\in\{1,\dots,n\}$ there is a greatest ordinal ε such that $\lambda(\eta^{\delta-1})\le\varepsilon\le\beta$ and $i\in J^{\varepsilon}$. By definition 3.1 we have $X_i^{\varepsilon} = X_i^{\varepsilon+1}=\dots=X_i^{\beta}$ with $X_i^{\varepsilon}=F_i(Z^1,\dots,Z^m)$. Since $\varepsilon\ge\lambda(\eta^{\delta-1})$ we have $\forall j\in\{1,\dots,m\}$, $\forall k\in\{1,\dots,n\}$, $\eta^{\delta-1}\le(S_j^{\varepsilon})_k<\varepsilon$. Therefore by induction hypothesis $B_k^{\delta-1}\sqsubseteq X_k^{(S_j^{\varepsilon})_k}$ and by definition 3.1 $Z_k^j=X_k^{(S_j^{\varepsilon})_k}$. By transitivity and definition of the ordering $\sqsubseteq$ of L^n we know that $\forall j\in\{1,\dots,m\}$, $B^{\delta-1}\sqsubseteq Z^j$ so that by definition 3.3.8 and monotony we get $B_i^{\delta} = F_i(\sigma(B^{\delta-1})) \sqsubseteq F_i(Z^1,\dots,Z^m) = X_i^{\varepsilon} = X_i^{\beta}$.
Finally $B^{\delta}\sqsubseteq X^{\eta^{\delta}}$.

Case 2.2. Assume that β is a successor ordinal and that for every $\eta^{\delta}\le\alpha<\beta$ we have $B^{\delta}\sqsubseteq X^{\alpha}$. We prove $\forall i=1,\dots,n$ $B_i^{\delta}\sqsubseteq X_i^{\beta}$. If $i\notin J^{\beta}$ then $B_i^{\delta}\sqsubseteq X_i^{\beta-1}=X_i^{\beta}$ else $i\in J^{\beta}$ and $X_i^{\beta}=F_i(Z^1,\dots,Z^m)$. Since $\beta>\eta^{\delta}>\Pi\lambda(\eta^{\delta-1})\ge\lambda(\eta^{\delta-1})$ we know that $\forall j\in\{1,\dots,m\}$, $\forall k\in\{1,\dots,n\}$ we have $\eta^{\delta-1}\le(S_j^{\delta})_k$ so that $B_k^{\delta-1}\sqsubseteq X_k^{(S_j^{\beta})_k}$ by induction hypothesis when $\delta'=\delta-1$. Therefore $\forall j\in\{1,\dots,m\}$ we have $B^{\delta-1}\sqsubseteq Z^j$ so that by monotony
$B_i^{\delta} = F_i(\sigma(B_i^{\delta-1})) \sqsubseteq F_i(Z^1,\dots,Z^m) = X_i^{\beta}$.

Case 2.3. Assume that β is a limit ordinal and that for every $\eta^{\delta}\le\alpha<\beta$ we have $B^{\delta}\sqsubseteq X^{\alpha}$. This implies $B^{\delta} \sqsubseteq \bigsqcup_{\eta^{\delta}\le\alpha<\beta} X^{\alpha} \sqsubseteq \bigsqcup_{\alpha<\beta} X^{\alpha} = X^{\beta}$

By transfinite induction on β (cases 2.1, 2.2, 2.3) we have proved that if the lemma is true for every $\delta'<\delta$ and δ is a successor ordinal then it is also true for δ.

Case 3. Assume that δ is a limit ordinal and that the lemma is true for every $\delta'<\delta$. We prove by transfinite induction on β that $\forall\beta\ge\eta^{\delta}$, $B^{\delta}\sqsubseteq X^{\beta}$.

Case 3.1. If $\beta=\eta^{\delta}$ then by lemma 3.3.4 β is also a limit ordinal. By induction hypothesis on δ we have $\forall\gamma<\delta$, $B^{\gamma}\sqsubseteq X^{\eta^{\gamma}}$ therefore $B^{\delta} = \bigsqcup_{\gamma<\delta} B^{\gamma} \sqsubseteq \bigsqcup_{\gamma<\delta} X^{\eta^{\gamma}}$. By lemma 3.3.3 the sequence $<\eta^{\gamma},\gamma\in Ord>$ is strictly increasing $\{\gamma<\delta \Rightarrow \eta^{\gamma}<\eta^{\delta}\}$. Therefore $\bigsqcup_{\gamma<\delta} X^{\eta^{\gamma}} \sqsubseteq \bigsqcup_{\eta^{\gamma}<\eta^{\delta}} X^{\eta^{\gamma}}$ and $\bigsqcup_{\eta^{\gamma}<\eta^{\delta}} X^{\eta^{\gamma}} \sqsubseteq \bigsqcup_{\alpha<\eta^{\delta}} X^{\alpha} = X^{\eta^{\delta}}$ so that by transitivity we conclude $B^{\delta}\sqsubseteq X^{\beta}$.

Case 3.2. Assume that β is a successor ordinal and that for every $\eta^{\delta}\leq\alpha<\beta$ we have $B^{\delta}\sqsubseteq X^{\alpha}$. We prove $\forall i=1,\ldots,n$ $B_i^{\delta}\sqsubseteq X_i^{\beta}$. If $i\notin J^{\beta}$ then $B_i^{\delta}\sqsubseteq X_i^{\beta-1}=X_i^{\beta}$ else $i\in J^{\beta}$ and $X_i^{\beta}=F_i(Z^1,\ldots,Z^m)$. Since β is strictly greater than the limit ordinal η^{δ} conditions 3.1(b)-(d) imply that $\forall j\in\{1,\ldots,m\}$, $\forall k\in\{1,\ldots,n\}$, $\eta^{\delta}\leq(S_j^{\beta})_k<\beta$. Therefore by induction hypothesis $\forall j\in\{1,\ldots,m\}$, $B_k^{\delta}\sqsubseteq X_k^{(S_j^{\beta})_k}$ so that $B^{\delta}\sqsubseteq Z^j$. Since the sequence $<B^{\delta},\delta\in Ord>$ is increasing (Cousot[3]) we get by monotony $B_i^{\delta}\sqsubseteq F_i(\sigma(B^{\delta})) \sqsubseteq F_i(Z^1,\ldots,Z^m) = X_i^{\beta}$.

Case 3.3. Assume that β is a limit ordinal and that for every $\eta^{\delta}\leq\alpha<\beta$ we have $B^{\delta}\sqsubseteq X^{\alpha}$. This implies $B^{\delta} \sqsubseteq \bigsqcup_{\eta^{\delta}\leq\alpha<\beta} X^{\alpha} \sqsubseteq \bigsqcup_{\alpha<\beta} X^{\alpha} = X^{\beta}$ so that by transitivity we conclude $B^{\delta}\sqsubseteq X^{\beta}$.

By transfinite induction on β (cases 3.1, 3.2, 3.3) we have proved that if lemma 3.3.9 is true for every $\delta'<\delta$ and δ is a limit ordinal then it is also true for δ.

By transfinite induction on δ (cases 1, 2, 3) we conclude $\forall\beta,\delta\in Ord$, $\{\beta\geq\eta^{\delta}\} \Rightarrow \{B^{\delta}\sqsubseteq X^{\beta}\}$. *End of Proof.*

CONVERGENCE THEOREM 3.3.10. An asynchronous iteration sequence with memory corresponding to the operator $F\in(L^n)^m\rightarrow L^n$ and starting with a prefixed point D of F is stationary, its limit is $\mathit{lis}(F\circ\sigma)(D)$.

Proof : We know (Cousot[3]) that the iteration sequence $<B^{\delta},\delta\in Ord>$ for $F\circ\sigma$ is stationary. Therefore there is an ordinal ε such that $\forall\gamma\geq\varepsilon$, $\mathit{lis}(F\circ\sigma)(D)=B^{\gamma}$. Hence by lemma 3.3.9 we have $\forall\beta\geq\eta^{\varepsilon}$, $\mathit{lis}(F\circ\sigma)(D) = B^{\varepsilon} \sqsubseteq X^{\beta}$ and by lemma 3.3.2 we have $X^{\beta} \sqsubseteq \mathit{lis}(F\circ\sigma)(D)$. By antisymmetry we conclude that $\forall\beta\geq\eta^{\varepsilon}$, $X^{\beta}=\mathit{lis}(F\circ\sigma)(D)$. *End of Proof.*

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I want to express my gratitude to François ROBERT for kindly drawing my attention to J.C. MIELLOU's result (*Algorithmes de relaxation : propriétés de convergence monotone*, Séminaire d'Analyse Numérique n°278, L.A.7, Université Scientifique et Médicale de Grenoble, Grenoble, France, June 1977) which was unknown to me. My results can be regarded as a generalization of those obtained by J.C. MIELLOU : in definition 2.1 (3.1) I do not use the additinal hypothesis $\{\forall i \in \{1,\dots n\},\ \forall \delta,\delta' \in \mathrm{Ord}, \{\delta \leq \delta'\} \Rightarrow \{S_i^{\delta} \leq S_i^{\delta'}\}$. This extra assumption facilitates very much the proof of the convergence theorem since it implies that the iteration sequence is an increasing chain wheras my main difficulty was to face the case of partially (but not totally) ordered iterates. By contrast this added supposition introduces a limitation in the scheduling policy. Also in order to prove that the iterates converge to a fixed point J.C. MIELLOU uses the hypotheses that either L is a finite ordered space or that L is a normal Banach lattice and F is semi-continuous. This last assumption is comparable to the continuity hypothesis that I discuss in the introduction and that I avoided since Tarski's theorem can be proved constructively [3] with a monotony hypothesis only.