2. PLANARITY TESTING

A <u>cut-vertex</u> or <u>cut-edge</u> of a graph is a vertex or edge whose deletion increases the number of connected components. A maximal connected subgraph of a graph that has no cut-vertex is called a <u>block</u>. In particular, every cut-edge is a block.

<u>Proposition 2.1.</u> Two blocks in a graph share at most one vertex, which must be a cut-vertex.

This leads to a cactus-like decomposition of a graph G:

All blocks of a graph can be found by <u>depth-first-search</u>, i.e., by exploring first the neighbors of the vertex v that was discovered last. If we find an unexplored neighbor w of v, we add the edge vw to the search tree T, and continue the search from w. Otherwise, we backtrack in T to the parent u of v. If no vertex of T below v has an ancestor above u, then the part below v together with u is the vertex set of a block.

In this case delete the part below v (including v) from the search tree T. At any rate, continue the search from u as long as v is not the point we started the search with (i.e., not the root of T).

A graph G is called <u>k-connected</u> if it cannot be disconnected by deleting $k-1$ vertices, and $|V(G)| \geq k$. Clearly, G is 2-connected if and only if it consists of a single block.

<u>Hw.</u> Prove that G is 2-connected if and only any pair of its vertices belong to a simple cycle.

This statement has a far-reaching generalization:

<u>Menger's Theorem</u> (1927)
A graph G is k-connected if and only if any pair of its vertices can be connected by k (internally) vertex-disjoint paths.

<u>Corollary</u> (Dirac 1960)
If G is k-connected $(k \geq 2)$, then any k vertices of G lie on a simple cycle.

Definition 2.2. An st-ordering of G is a numbering of its vertex set $v_1, v_2, \dots, v_n$ such that every v_i $(1 < i < n)$ has a neighbor v_j with $j < i$ and a neighbor v_k with $k > i$, furthermore we have $v_1 v_n \in E(G)$.

Proposition 2.3. G admits an st-ordering if and only if it is 2-connected. Moreover, in this case any pair of adjacent vertices can be chosen as v_1 and v_n (the "source" and the "sink").

For the proof we need

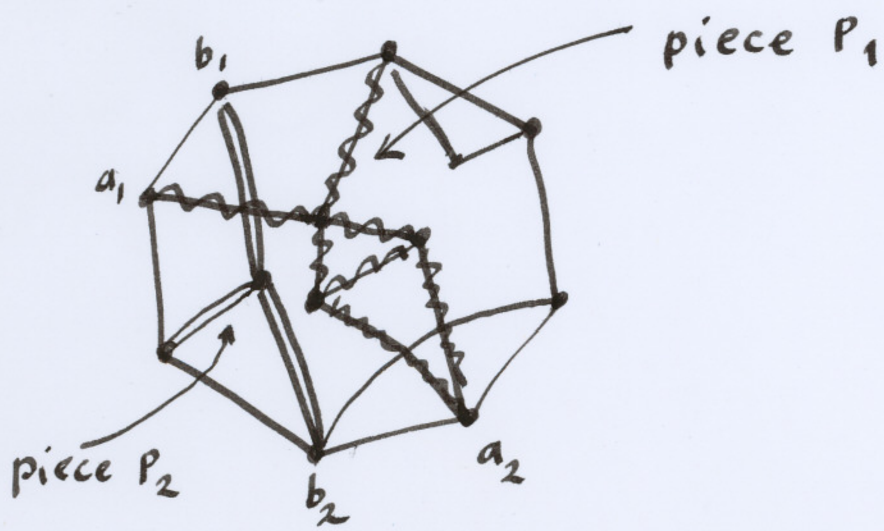
Lemma 2.4. (Ear decomposition lemma)
Every 2-connected graph can be obtained from a cycle by successively adding paths such that each path has both of its endpoints on the current graph, but otherwise it is internally disjoint from it.

Proof. Let $G_i \subsetneqq G$ be the current graph. Pick $u \in V(G_i)$, $v \notin V(G_i)$ such that $uv \in E(G)$ and connect v to G_i by a shortest path. $\square$

Proof of Proposition 2.3. Let G be 2-connected. We show by induction on $|E(G)|$ that it admits an st-numbering with $v_1 = u$, $v_n = v$, $uv \in E(G)$.

Pick a cycle C through $uv \in E(G)$. If $G = C$, there is nothing to prove. Otherwise add a path to it, as in the ear decomposition, and number its internal vertices so that they form an increasing chain connecting its endpoints. □

<u>Definition 2.5.</u> Given a cycle C in a 2-connected graph G, the edges connecting 2 vertices of C and the connected components of $G - C$ together with the edges connecting them to C are called <u>pieces</u> of G with respect to C. The vertices of a piece P that belong to C are called the <u>attachments</u> of P. The cycle C is <u>separating</u> if it gives rise to at least two pieces. Two pieces <u>interlace</u> (<u>conflict</u>) if they cannot be drawn on the same side of C, i.e., P_1 has two attachments a_1 and a_2, and P_2 has two attachments b_1 and b_2 such that their cyclic order is a_1, b_1, a_2, b_2.



Proposition 2.6. If C is a nonseparating cycle with a piece P that is not a path, then G also has a separating cycle. □

Proposition 2.7. Let G be a 2-connected graph with a cycle C. G is planar if and only if the following two conditions are satisfied.

(1) Each piece P with respect to C forms a planar graph together with C.

(2) The interlacement (conflict) graph of the pieces is bipartite (2-colorable). [The conflict graphs vertex set is the set of pieces with respect to C, two pieces being joined by an edge if and only if they interlace.]

Proof: Homework.

Based on Proposition 2.7, we can design a simple recursive algorithm for testing planarity of a 2-connected graph with n vertices (and $O(n)$ edges). Finding the pieces with respect to the cycle C takes $O(n)$ time. The conflict graph can be built in $O(n^2)$ time. The number of recursive calls is $O(n)$. Altogether, the running time in the simplest implementation is $O(n^3)$.