5. Other graph representations

In the spirit of Koebe's theorem, one can consider other representations of graphs by touching or intersecting geometric objects. An old open problem in this field is due to Scheinerman.

Problem 5.1. Can the vertices of every planar graph G be represented by line segments in the plane so that two segments have a point in common if and only if the corresponding vertices are adjacent in G?

For bipartite graphs, the answer is yes. Hartman, Newman, and Ziv / de Fraysseix, de Mendez, and Pach showed that in this case it is sufficient to use vertical and horizontal segments without introducing any proper crossing. De Fraysseix and de Mendez extended this result to 3-colorable planar graphs, but the 3 directions are not always sufficient. In fact, for general planar graphs we do not even know if a representation by continuous arcs ("strings") always exists, with the property that two arcs cannot intersect more than once.

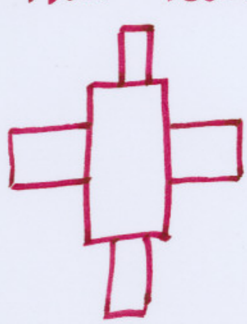
Problem 5.2. (Harborth) Does every planar graph admit a straight-line drawing in which the length of every edge is an integer.

Conjecture 5.3. (Papadimitriou-Ratajczak) Any 3-connected planar graph can be drawn in the plane with possibly crossing straight-line edges so that for any ordered pair of vertices (u,v), there is a path from u to v along which the Euclidean distance to v strictly decreases.

It is not hard to show that $K_{2,11}$ and $K_{3,16}$ cannot be drawn with the required property, but the former is not 3-connected and the latter is not planar.

Homework: Prove that $K_{2,11}$ cannot be drawn with the required property.

Returning to the representations by horizontal and vertical segments, we can obviously extend them into nonoverlapping axis-parallel rectangles,



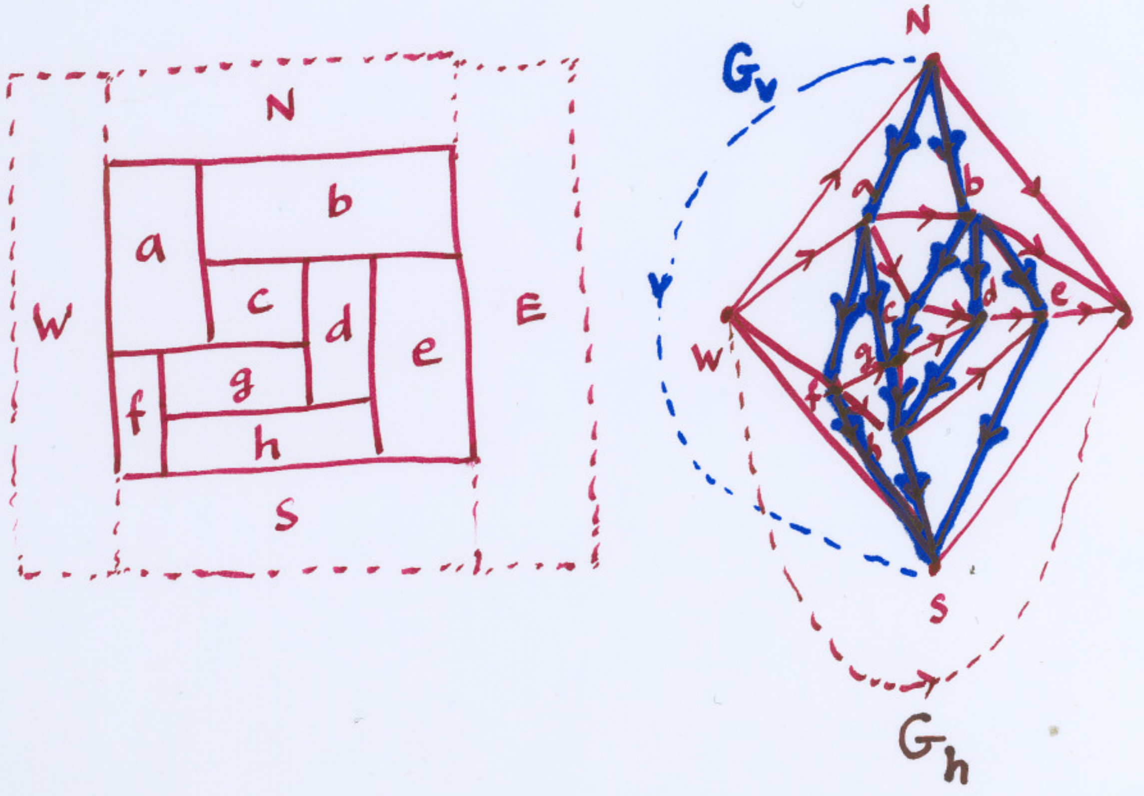
until each edge of a rectangle becomes tangent to the edge of another, and even further...

The question arises: Which graphs G admit a <u>rectangular dual</u>, that is, a representation of its vertices by axis-parallel rectangles that form a tiling of a rectangle such that

1. two rectangles are tangent if and only if the corresponding vertices are adjacent in G
2. no four rectangles share a vertex.

<u>Theorem 5.4</u> (Ungar, Bhasker-Sahni) A planar graph admits a rectangular dual if and only if

1. its exterior face is a quadrilateral;
2. its interior faces are triangles;
3. it has no separating triangle.



A graph G satisfying the conditions in Theorem 5.4 can be directed and split into two edge-disjoint subgraphs G_h and G_v (where the subscript "h" and "v" stand for horizontal and vertical) such that the counterclockwise order of their edges around each internal vertex is:

in-edges in G_h, out-edges in G_v,
out-edges in G_h, in-edges in G_v.

Looking at G_h (and G_v), we see that the in-edges and out-edges are separated around each point, and both graphs have only one source and one sink. Xin He developed several efficient algorithms for constructing two such directed subgraphs for any G satisfying the conditions, from where one can obtain a rectangular dual representation of G.

Denote the faces in G_h (G_v) separating the in- and out-edges at an internal vertex u by lower(u) and upper(u) (left(u) and right(u), respectively). For any face f in G_h (G_v), let $d(f)$ denote the length of the longest "monotone" chain of faces connecting the left-most (uppermost) face to f. Let the x-coordinates of the vertical sides of the rectangle that represents u be d(left(u)) and d(right(u)), and let the y-coordinates of the horizontal sides of this rectangle be d(upper(u)) and d(lower(u)). $\square$

Given a rectangular tiling (as in Theorem 5.4) with the property that no four rectangles share a corner, we can number the rectangles as follows. Let us give label 1 to the rectangle containing the lower left corner of the big rectangle R containing all small ones. Notice that one of the following two possibilities holds:

1. The right side of the rectangle labeled 1 is a full segment in the tiling;
2. The upper side of the rectangle labeled 1 is a full segment in the tiling (that is, its extension by a small amount would cut into another piece).

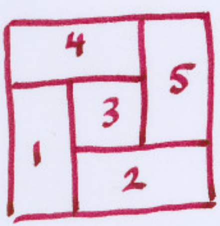
In the first case shrink rectangle 1 into a vertical segment on the boundary of R, in the second case shrink it into a horizontal one. Now another inner rectangle will contain the lower left corner of R; let's label it with 2. We continue this procedure until we run out of rectangles, i.e., all internal rectangles get labeled.

Now follow the same "shrinking procedure" at the upper left corner of R, and write down the sequence of the labels of the internal rectangles as they disappear. In this way we obtain a permutation of the labels. Ackerman et al. showed that this permutation uniquely determines the combinatorial type of the tiling.

<u>Lemma 5.5.</u> Let $i < j$ be two labels.

(i) If i precedes j in the permutation, then rectangle i is completely to the left of rectangle j.

(ii) I j precedes i in the permutation, then rectangle i is completely below rectangle j. $\square$



$\Rightarrow$ permutation 4 1 3 5 2