Introduction to Combinatorics

Midterm Test November 7, 2001

1. Let $f(n)$ denote the largest number of unordered triples one can choose from the set $\{1, 2, \ldots, n\}$ so that any two triples have precisely two elements in common. Determine or estimate $f(n)$.

2. In how many different ways can one choose 5 numbers from the set $\{1, 2, \ldots, 20\}$ so that no two of them are consecutive?

3. An architect is standing in a freshly finished apartment and holding a floor plan of the same apartment horizontally in his hands. Prove that some point in the plan is positioned exactly above the point on the apartment's floor it corresponds to.

Does the statement remain true if the map is not held horizontally?

4. Let $\mathcal{I}$ be a system of intervals along a line, all of whose endpoints are distinct. For any two elements $I, J \in \mathcal{I}$, we write $I \leq J$ if I lies entirely to the left of J.

(i) Show that $(\mathcal{I}, \leq)$ is a partial ordering.

(ii) What statement do we obtain if we apply Dilworth's theorem to $(\mathcal{I}, \leq)$?

5. Prove the following identity:

$$\binom{n}{k}\binom{k}{l} = \binom{n}{l}\binom{n-l}{k-l}.$$

Bonus question: Let $A_1, A_2, \ldots, A_m$ be distinct nonempty subsets of an n-element set with the property that any two A_i's are either disjoint or one of them contains the other. Prove that $m \leq 2n - 1$. (For extra credit!)

Please explain all of your answers! Good luck! - J.P.