A computational approach to Conway's thrackle conjecture

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Abstract

A drawing of a graph in the plane is called a *thrackle* if every pair of edges meets precisely once, either at a common vertex or at a proper crossing. Let $t(n)$ denote the maximum number of edges that a thrackle of n vertices can have. According to a 50 years old conjecture of Conway, $t(n) = n$ for every $n \geq 3$. For any $\varepsilon > 0$, we give an algorithm terminating in $e^{O((1/\varepsilon^2)\ln(1/\varepsilon))}$ steps to decide whether $t(n) \leq (1+\varepsilon)n$ for all $n \geq 3$. Using this approach, we improve the best known upper bound, $t(n) \leq \frac{3}{2}(n-1)$, due to Cairns and Nikolayevsky, to $\frac{167}{117}n < 1.428n$.

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1 Introduction

A *drawing* of a graph (or a *topological graph*) is a representation of the graph in the plane such that the vertices are represented by distinct points and the edges by (possibly crossing) simple continuous curves connecting the corresponding point pairs and not passing through any other point representing a vertex. If it leads to no confusion, we make no notational distinction between a drawing and the underlying abstract graph G. In the same vein, $V(G)$ and $E(G)$ will stand for the vertex set and edge set of G as well as for the sets of points and curves representing them.

A drawing of G is called a *thrackle* if every pair of edges meet precisely once, either at a common vertex or at a proper crossing. (A crossing p of two curves is *proper* if at p one curve passes from one side of the other curve to its other side.) More than *forty* years ago Conway [18, 2, 15] conjectured that every thrackle has at most as many edges and vertices, and offered a bottle of beer for a solution. Since then the prize went up to a thousand dollars. In spite of considerable efforts, Conway's thrackle conjecture is still open. It is believed to represent the tip of an "iceberg," obstructing our understanding of crossing patterns of edges in topological graphs. If true, Conway's conjecture would be tight as any cycle of length at least *five* can be drawn as a thrackle, see [17]. Two thrackle drawings of C_5 and C_6 are shown in Figure 1.

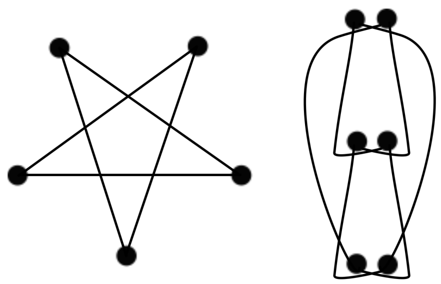

Figure 1: C_5 and C_6 drawn as thrackles

Obviously, the property that G can be drawn as a thrackle is *hereditary*: if G has this property, then any subgraph of G does. It is very easy to verify (cf. [17]) that C_4, a cycle of length *four*, cannot be drawn in a thrackle. Therefore, every "thrackleable" graph is C_4-free, and it follows from extremal graph theory that every thrackle of n vertices has at most $O(n^{3/2})$ edges [6]. The first linear upper bound on the maximum number of edges of a thrackle of n vertices was given by Lovász et al. [12]. This was improved to a $\frac{3}{2}(n-1)$ by Cairns and Nikolayevsky [3].

The aim of this note is to provide a finite approximation scheme for estimating the maximum number of edges that a thrackle of n vertices can have. We apply our technique to improve the best known upper bound for this maximum.

To state our results, we need a definition. Given three integers $c', c'' > 2$, $l \geq 0$, the *dumbbell* $\mathrm{DB}(c', c'', l)$ is a simple graph consisting of two disjoint cycles of length c' and c'', connected by a path of length l. For $l = 0$, the two cycles share a vertex. It is natural to extend this definition to negative values of l, as follows. For any $l > -\min(c', c'')$, let $\mathrm{DB}(c', c'', l)$ denote the graph consisting of two cycles of lengths c' and c'' that share a path of length $-l$. That is, for any $l > -\min(c', c'')$, we have

$$|V(\mathrm{DB}(c', c'', l))| = c' + c'' + l - 1.$$

The three types of dumbbells (for $l < 0$, $l = 0$, and $l > 0$) are illustrated in Figure 2.

Figure 2: Dumbbells $\mathrm{DB}(6,6,-1), \mathrm{DB}(6,6,0)$, and $\mathrm{DB}(6,6,1)$

Our first theorem shows that for any $\varepsilon > 0$, it is possible to prove Conway's conjecture up to a multiplicative factor of $1+\varepsilon$, by verifying that no dumbbell smaller than a certain size depending on ε is thrackleable.

Theorem 1. *Let $c \geq 6$ and $l \geq -1$ be two integers with the property that no dumbbell $\mathrm{DB}(c', c'', l')$ with $-c'/2 \leq l' \leq l$ and with even $6 \leq c', c'' \leq c$ can be drawn in the plane as a thrackle. Let $r = \lfloor l/2 \rfloor$. Then the maximum number of edges $t(n)$ that a thrackle on n vertices can have satisfies $t(n) \leq \tau(c,l)n$, where*

$$\tau(c,l) = \begin{cases} \frac{47c^2+116c+80}{35c^2+68c+32} & \text{if } l = -1, \\ 1 + \frac{2c^2r+4cr^2+22cr+7c^2+22c+8r^2+24r+16}{2c^2r^2+14c^2r+4cr^2+16cr+24c^2+12c} & \text{if } l \geq 0, \end{cases}$$

as n tends to infinity.

As both c and l get larger, the constant $\tau(c,l)$ given by the second part of Theorem 1 approaches 1. On the other hand, assuming that Conway's conjecture is true for all bipartite graphs with up to 10 vertices, which will be verified in Section 4, the first part of the theorem applied with $c = 6, l = -1$ yields that $t(n) \leq \frac{617}{425}n < 1.452n$. This bound is already better than the bound $\frac{3}{2}n$ established in [3].

By a more careful application of Theorem 1, we obtain an even stronger result.

Theorem 2. *The maximum number of edges $t(n)$ that a thrackle on n vertices can have satisfies the inequality $t(n) \leq \frac{167}{117}n < 1.428n$.*

Our method is algorithmic. We design an $e^{O((1/\varepsilon^2)\ln(1/\varepsilon))}$ time algorithm to prove, for any $\varepsilon > 0$, that $t(n) \leq (1+\varepsilon)n$ for all n, or to exhibit a counterexample to Conway's conjecture. The proof of Theorem 2 is computer assisted: it requires testing the planarity of certain relatively small graphs.

For thrackles drawn by straight-line edges, Conway's conjecture had been settled in a slightly different form by Hopf and Pannwitz [10] and by Sutherland [16] well before Conway was born, and later, in the above form, by Erdős and Perles. Assuming that Conway's conjecture is true, Woodall [17] gave a complete characterization of all graphs that can be drawn as a thrackle. He also observed that it would be sufficient to verify the conjecture for dumbbells. This observation is one of the basic ideas behind our arguments.

Several interesting special cases and variants of the conjecture are discussed in [3, 4, 5, 9, 12, 13, 14].

In Section 2, we describe a crucial construction of Conway and summarize some earlier results needed for our arguments. The proofs of Theorems 1 and 2 are given in Sections 3 and 4. The analysis of the algorithm for establishing the $(1+\varepsilon)n$ upper bound for the maximum number of edges that a thrackle of n vertices can have is also given in Section 4 (Theorem 7). In the last section, we discuss some related Turán-type extremal problems for planar graphs.

2 Conway's doubling and preliminaries

In this section, we review some earlier results that play a key role in our arguments.

A *generalized thrackle* is a drawing of a graph in the plane with the property that any pair of edges share an odd number of points at which they properly cross or which are their common endpoints. Obviously, every thrackle is a generalized thrackle but not vice versa: although C_4 is not thrackleable, it can be drawn as a generalized thrackle, which is not so hard to see.

We need the following simple observation based on the Jordan curve theorem.

Lemma 3. [12] *A (generalized) thrackle cannot contain two vertex disjoint odd cycles.*

Lovász, Pach, and Szegedy [12] gave a somewhat counterintuitive characterization of generalized thrackles containing no odd cycle: a bipartite graph is a generalized thrackle if and only if it is *planar*. Moreover, it follows immediately from Lemma 3 and the proof of Theorem 3 in Cairns and Nikolayevsky [3] that this statement can be strengthened as follows.

Lemma 4. [3] *Let G be a bipartite graph with vertex set $V(G) = A \cup B$ and edge set $E(G) \subseteq A \times B$. If G is a generalized thrackle then it can be redrawn in the plane without crossing so that the cyclic order of the edges around any vertex $v \in V(G)$ is preserved if $v \in A$ and reversed if $v \in B$.*

We recall a construction of Conway for transforming a thrackle into another one. It can be used to eliminate odd cycles.

Let G be a thrackle or a generalized thrackle which contains an *odd* cycle C. In the literature, the following procedure is referred to as *Conway's doubling*: First, delete from G all edges incident to at least one vertex belonging to C, including all edges of C. Replace every vertex v of C by two nearby vertices, v_1 and v_2. For any edge vv' of C, connect v_1 to v'_2 and v_2 to v'_1 by two edges running very close to the original edge vv', as depicted in Figure 3. For any vertex v belonging to C, the set of edges incident to v but not belonging to C can be divided into two classes, $E_1(v)$ and $E_2(v)$: the sets of all edges whose initial arcs around v lie on one side or the other side of C. In the resulting topological graph G', connect all edges in $E_1(v)$ to v_1 and all edges in $E_2(v)$ to v_2 so that every edge connected to v_1 crosses all edges connected to v_2 exactly once in their small neighborhood. See Figure 3. All other edges of G remain unchanged. Denote the vertices of the original odd cycle C by $v^1, v^2, \ldots, v^k$, in this order. In the resulting drawing G', we obtain an *even* cycle $C' = v_1^1 v_2^2 v_1^3 v_2^4 \ldots v_2^1 v_1^2 v_2^3 v_1^4 \ldots$ instead of C. It is easy to verify that G' is drawn as a thrackle, which is stated as part (ii) of the following lemma (see also Lemma 2 in [3]).

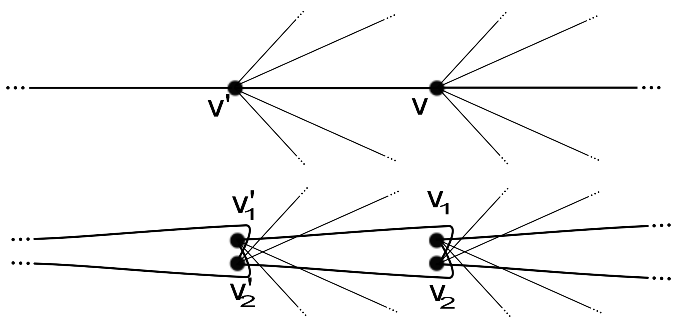

Figure 3: Conway's doubling of a cycle

Lemma 5. (Conway, [17, 3]) *Let G be a (generalized) thrackle with at least one odd cycle C. Then the topological graph G' obtained from G by Conway's doubling of C is*

(i) bipartite, and

(ii) a (generalized) thrackle.

Proof. It remains to verify part (i). Let k denote the length of the (odd) cycle $C \subseteq G$, and let C' stand for the doubled cycle in G'. The length of C' is $2k$. Let π denote the inverse of the doubling transformation. That is, π identifies the opposite pairs of vertices in C', and takes C' into C.

Suppose for a contradiction that G' is not bipartite. In view of Lemma 3, no odd cycle of G' is disjoint from C'. Let D' be an odd cycle in G' with the smallest number of edges that do not belong to C'. We can assume that D' is the union of two paths, P_1 and P_2, connecting the same pair of vertices u, v in C', where P_1 belongs to C' and P_2 has no interior points on C'.

If $\pi(u) \neq \pi(v)$, that is, the length of P_1 is not 0 or k, then $\pi(D') = \pi(P_1) \cup \pi(P_2)$ is a simple cycle in G. Notice that the lengths of P_1 and P_2 have different parities. If the length of P_1 is even, say, then, according to the rules of doubling, the initial and final pieces of P_2 in small neighborhoods of u and v are on the same side of the (arbitrarily oriented) cycle C'. Consequently, the initial and final pieces of $\pi(P_2)$ in small neighborhoods of $\pi(u)$ and $\pi(v)$ are on the *same* side of C. On the other hand, using the fact that G is a generalized thrackle, the total number of intersection points between the odd path $\pi(P_2)$ and the odd cycle C is odd (see the proof of Lemma 2.2 from [12]). This yields that the initial and final pieces of $\pi(P_2)$ in small neighborhoods of $\pi(u)$ and $\pi(v)$ must lie on *different* sides of C, a contradiction.

The cases when P is odd and when $\pi(u) = \pi(v)$ can be treated analogously. □

Finally, we recall an observation of Woodall [17] mentioned in the introduction, which motivated our investigations.

As thrackleability is a hereditary property, a minimal counterexample to the thrackle conjecture must be a connected graph G with exactly $|V(G)|+1$ edges and with no vertex of degree *one*. Such a graph G is necessarily a dumbbell $\mathrm{DB}(c', c'', l)$. If $l \neq 0$, then G consists of two cycles that share a path or are connected by a path uv. In both cases, we can "double" the path uv, as indicated in Figure 4, to obtain another thrackle G'. It is easy to see that G' is a dumbbell consisting of two cycles that share precisely one vertex. Moreover, if any of these two cycles is not even, then we can double it and repeat the above procedure, if necessary, to obtain a dumbbell $\mathrm{DB}(b', b'', 0)$ drawn as a thrackle, where b' and b'' are even numbers.

Thus, in order to prove the thrackle conjecture, it is enough to show that no dumbbell $\mathrm{DB}(c', c'', 0)$ consisting of two even cycles that share a vertex is thrackleable.

3 Proof of Theorem 1

Let $c \geq 6$ and $l \geq -1$ be two integers, and suppose that no dumbbell $\mathrm{DB}(c', c'', l')$ with $-c'/2 \leq l' \leq l$ and with even $6 \leq c', c'' \leq c$ can be drawn in the plane as a thrackle. For simpler notation, let $r = \lfloor l/2 \rfloor$.

Let $G = (V, E)$ be a thrackleable graph with n vertices and m edges. We assume without loss of generality that G is connected and that it has no vertex of degree *one*. Otherwise, we can successively delete all vertices of degree *one*, and argue for each connected component of the resulting graph separately.

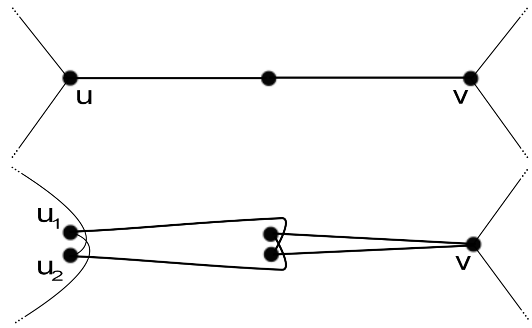

Figure 4: Doubling the path uv

As usual, we call a graph *two-connected* if it is connected and it has no *cutvertex*, i.e., it cannot be separated into two or more parts by the removal of a vertex [6].

We distinguish three cases:

(A) G is bipartite;

(B) G is not bipartite, and the graph G' obtained by performing Conway's doubling of a shortest odd cycle $C \subset G$ is 2-connected;

(C) G is not bipartite, and the graph G' obtained by performing Conway's doubling of a shortest odd cycle $C \subset G$ is not 2-connected.

In each case, we will prove that $m \leq \tau(c, l)n$.

(A) By Lemma 4, in this case G is planar. We fix an embedding of G in the plane. According to the assumption of our theorem, G contains no subgraph which is a dumbbell $\mathrm{DB}(c', c'', l')$, for any even $6 \leq c' \leq c'' \leq c$, and $-c'/2 \leq l' \leq l$. We also know that G has no C_4. We are going to use these conditions to bound the number of edges $m = |E(G)|$.

Notice that it is unnecessary to exclude dumbbells $\mathrm{DB}(c', c'', l')$ with $l' < -c'/2$, because for such values of the parameters, exchanging the roles of two arcs of the cycle of length c', we obtain that $\mathrm{DB}(c', c'', l') = \mathrm{DB}(c', c' + c'' + 2l', -c' - l')$ with $c' + c'' + 2l' < c'' \leq c$ and $\max(-c'/2, -(c' + c'' + 2l')/2) \leq -c' - l' < 0$. A similar argument shows that there is no need to exclude pairs of cycles such that the edges shared by them do not form a connected arc along one of them.

Suppose first that G is *two-connected.* Let f denote the number of faces, and let f_c stand for the number of faces with at most c sides. By double counting the edges, we obtain

$$2m \geq 6f_c + (c + 2)(f - f_c). \tag{1}$$

If $l = -1$, then applying the condition on forbidden dumbbells, we obtain that no two faces of size at most c share an edge, so that $6f_c \leq m$. If $l \geq 0$, Menger's theorem implies that any two faces of size at most c are connected by two vertex disjoint paths. Since any such path must be longer than l, to each face we can assign its vertices as well as the $r = \lfloor l/2 \rfloor$ closest vertices along two vertex disjoint paths leaving the face, and these sets are disjoint for distinct faces. Thus, we have $f(2r + 6) \leq n$. In either case, we have

$$f_c \leq \begin{cases} \frac{m}{6} & \text{if } l = -1, \\ & \\ \frac{n}{2r+6} & \text{if } l \geq 0. \end{cases} \tag{2}$$

Combining the last two inequalities, we obtain

$$f \leq \frac{(c-4)f_c + 2m}{c+2} \leq \begin{cases} \frac{(c-4)\frac{m}{6}+2m}{c+2} & \text{if } l = -1, \\ & \\ \frac{(c-4)\frac{n}{2r+6}+2m}{c+2} & \text{if } l \geq 0. \end{cases}$$

In view of Euler's polyhedral formula $m + 2 = n + f$, which yields

$$m \leq \begin{cases} \frac{6c+2}{5c+4}n - \frac{12c+24}{5c+4} & \text{if } l = -1, \\ & \\ \frac{2cr+4r+7c+8}{2cr+6c}n - \frac{2cr+4r+6c+12}{cr+3c} & \text{if } l \geq 0. \end{cases} \tag{3}$$

It can be shown by routine calculations that the last estimates, even if we ignore their negative terms independent of n, are stronger than the ones claimed in the theorem. (In fact, they are also stronger than the corresponding bounds (5) and (4) in Case (B); see below.)

If G is not 2-connected, then consider a block decomposition of G, and proceed by induction on the number of blocks. The induction goes through, because the negative terms in (3), which are independent of n, are smaller than -2.

(B) In this case, we establish two upper bounds on the maximum number of edges in G: one that decreases with the length of the shortest odd cycle $C \subseteq G$ and one that increases. Finally, we balance between these two bounds.

By doubling a shortest odd cycle $C \subseteq G$, as before, we obtain a bipartite thrackle G' (see Lemma 5). Let C' denote the doubled cycle in G'. By Lemma 4, G' is a two-colorable planar graph. Moreover, it can be embedded in the plane without crossing so that the cyclic order of the edges around each vertex in one color class is preserved, and for each vertex in the other color class reversed. A closer inspection of the way how we double C shows that as we traverse C' in G', the edges incident to C' start on alternating sides of C'. This implies that, after redrawing G' as a plane graph, all edges incident to C' lie on one side, that is, C' is a *face*.

Slightly abusing the notation, from now on let G' denote a crossing-free drawing with the above property, which has a $2|C|$-sided face C'. Denoting the number of vertices and edges of G' by n' and m', the number of faces and the number of faces of size at most c by f' and f'_c, respectively, we have $n' = n + |C| = |V(G')|$, $m' = m + |C| = |E(G')|$, and, as in Case (A), inequality (2),

$$f'_c \leq \begin{cases} \frac{1}{6}m' & \text{if } l = -1, \\ & \\ \frac{n'}{2r+6} & \text{if } l \geq 0. \end{cases}$$

Double counting the edges of G', we obtain

$$2m' \geq 6f'_c + (c+2)(f' - 1 - f'_c) + 2|C|.$$

In case $l \geq 0$, combining the last two inequalities, we have

$$f \leq \frac{(c-4)f'_c + 2(m' - |C|) + c + 2}{c+2} \leq \frac{(c-4)\frac{n'}{2r+6} + 2(m' - |C|) + c + 2}{c+2}.$$

By Euler's polyhedral formula, $f' = m' - n' + 2$. Thus, after ignoring the negative term, which depends only on c and l, the last inequality yields

$$|E(G)| \leq \frac{2cr + 4r + 7c + 8}{2cr + 6c} n + |C| \frac{c-4}{2cr + 6c}. \quad (4)$$

The case $l = -1$ can be treated analogously, and the corresponding bound on $E(G)$ becomes

$$|E(G)| \leq \frac{6c + 12}{5c + 4} n + |C| \frac{c-4}{5c+4}. \quad (5)$$

We now establish another upper bound on the number of edges in G: one that decreases with the length of the shortest odd cycle C in G. As in [12], we remove from G the vertices of C together with all edges incident to them. Let G'' denote the resulting thrackle. By Lemma 3, G'' is bipartite. By Lemma 4, it is a planar graph. From now on, let G'' denote a fixed (crossing-free) embedding of this graph. According to our assumptions, G'' has no subgraph isomorphic to $\mathrm{DB}(c', c'', l')$, for any even numbers c' and c'' with $6 \leq c' \leq c'' \leq c$, and for any integer l' with $-c'/2 \leq l' \leq l$.

We can bound $|E(G'')|$, as follows. By the minimality of C, each vertex $v \in V(G)$ that does not belong to C is joined by an edge of G to at most one vertex on C. Indeed, otherwise, w would create either a C_4 or an odd cycle shorter than C. Hence, if $l \geq 0$, inequality (3) implies that

$$|E(G)| \leq |E(G'')| + |C| + (n - |C|) \leq \frac{2cr + 4r + 7c + 8}{2cr + 6c}(n - |C|) + n. \quad (6)$$

In the case $l = -1$, we obtain

$$|E(G)| \leq |E(G'')| + |C| + (n - |C|) \leq \frac{6c+2}{5c+4}(n - |C|) + n. \quad (7)$$

It remains to compare the above upper bounds on $|E(G)|$ and to optimize over the value of $|C|$. If $l > -1$, then the value of $|C|$ for which the right-hand sides of (4) and (6) coincide is

$$|C| = \frac{2cr + 6c}{2cr + 4r + 8c + 4} n.$$

The claimed bound follows by plugging this value into (4) or (6).

In the case $l = -1$, the critical value of $|C|$, obtained by comparing the bounds (5) and (7), is

$$|C| = \frac{5c + 4}{7c + 8} n.$$

Plugging this value into (5) or (7), the claimed bound follows.

(C) As before, let C be a shortest odd cycle in G, and let G' be the graph obtained from G after doubling C. The doubled cycle is denoted by $C' \subset G'$. Let $G_0 \supseteq C$ denote a *maximal* subgraph of G, which is turned into a *two*-connected subgraph of G' after performing Conway's doubling on C. Let G_1 stand for the graph obtained from G by the removal of all *edges* in G_0.

It is easy to see that G_1 is bipartite, and each of its connected components shares exactly one vertex with G_0. Indeed, if a connected component $G_2 \subseteq G_1$ were not bipartite, then, by Lemma 3, G_2 would share at least one vertex with C, which belongs to an odd cycle of G_2. By the maximal choice of G_0, after doubling C, the component G_2 must turn into a subgraph $G_2' \subset G'$, which shares precisely *one* vertex with the doubled cycle C'. Thus, G_2 must also share precisely *one* vertex with C, which implies that $G_2' \subseteq G'$ has an odd cycle. This contradicts Lemma 5(i), according to which G' is a bipartite graph.

Therefore, G_1 is the union of all blocks of G, which are not entirely contained in G_0. Since each connected component G_2 of G_1 is bipartite, the number of edges of G_2 can be bounded from above by (3), just like in Case (A).

In order to bound the number of edges of G, we proceed by adding the connected components of G_1 to G_0, one by one. As at the end of the discussion of Case (3), the fact that the last terms in (3), which do not depend on n, are smaller than -2, we can complete the proof by induction on the number of connected components of G_1.

4 A better upper bound

As was pointed out in the Introduction, if we manage to prove that for any l', $-3 \leq l' \leq -1$, the dumbbell $\mathrm{DB}(6,6,l')$ is not thrackleable, then Theorem 1 yields that the maximum number of edges that a thrackle on n vertices can have is at most $\frac{617}{425}n < 1.452n$. This estimate is already better than the currently best known upper bound $\frac{3}{2}n$ due to Cairns and Nikolayevsky [3].

In order to secure this improvement, we have to exclude the subgraphs $\mathrm{DB}(6,6,-1)$, $\mathrm{DB}(6,6,-2)$, and $\mathrm{DB}(6,6,-3)$. The fact that $\mathrm{DB}(6,6,-3)$ cannot be drawn as a thrackle was proved in [12] (Theorem 5.1). Here we present an algorithm that can be used for checking whether a “reasonably” small graph G can be drawn as a thrackle. We applied our algorithm to verify that $\mathrm{DB}(6,6,-1)$ and $\mathrm{DB}(6,6,-2)$ are indeed not thrackleable. In addition, we show that $\mathrm{DB}(6,6,0)$ cannot be drawn as thrackle, which leads to the improved bound in Theorem 2.

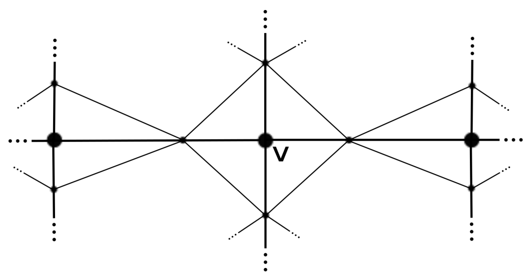

Figure 5: 4-cycle around a vertex v of G', which was a crossing point in G

Let $G = (V,E)$ be a thrackle. Direct the edges of G arbitrarily. For any $e \in E$, let $E_e \subseteq E$ denote the set of all edges of G that do not share a vertex with e, and let $m(e) = |E_e|$. Let $\pi_e = (\pi_e(1), \pi_e(2), \ldots, \pi_e(m(e)))$ stand for the $m(e)$-tuple (permutation) of all edges belonging to E_e, listed in the order of their crossings along e.

Construct a planar graph G' from G, by introducing a new vertex at each crossing between a pair of edges of G, and replacing each edge by its pieces. In order to avoid that G' has an embedding in which two paths corresponding to a crossing pair of edges of G do not *properly* cross, we introduce a new vertex in the interior of every edge of G', whose both endpoints are former crossings. For each former crossing point v, we add a cycle of length *four* to G', connecting its neighbors in their cyclic order around v, as illustrated in Figure 5. In the figure, the thicker lines and points represent edges and vertices or crossings of G, while the thinner lines and points depict the *four*-cycles added at the second stage.

Obviously, G' is completely determined by the directed abstract underlying graph of G and by the set of permutations $\Pi(G) := \{\pi_e \in E_e^{m(e)} | e \in E\}$. Thus, a graph $G = (V, E)$ can be drawn as a thrackle if and only if there exists a set Π of $|E|$ permutations of $E_e, e \in E$, such that the abstract graph G' corresponding to the pair (G, Π) is *planar*. In other words, to decide whether a given abstract graph $G = (V, E)$ can be drawn as a thrackle, it is enough to consider all possible sets of permutations Π of $E_e, e \in E$, and to check if the corresponding graph $G' = G'(G, \Pi)$ is planar for at least one of them. The first deterministic linear time algorithm for testing planarity was found by Hopcroft and Tarjan [11]. However, in our implementation we used an improved algorithm for planarity testing by Fraysseix et al. [7], in particular, its implementation in the library P.I.G.A.L.E. [8]. We leave the pseudocode of our routine for the abstract.

It was shown in [12] (Lemma 5.2) that in every drawing of a directed cycle C_6 as a thrackle, either every oriented path $e_1e_2e_3e_4$ is drawn in such a way that $\pi_{e_1} = (e_4, e_3)$ and $\pi_{e_4} = (e_1, e_2)$, or every oriented path $e_1e_2e_3e_4$ is drawn in such a way that $\pi_{e_1} = (e_3, e_4)$ and $\pi_{e_4} = (e_2, e_1)$. Using this observation (which is not crucial, but saves computational time), we ran a backtracking algorithm to rule out the existence of a set of permutations Π, for which $G'(\mathrm{DB}(6, 6, 0), \Pi)$, $G'(\mathrm{DB}(6, 6, -1), \Pi)$, or $G'(\mathrm{DB}(6, 6, -2), \Pi)$ is planar. Our algorithm attempts to construct larger and larger parts of a potentially good set Π, and at each step it verifies if the corresponding graph still has a chance to be extended to a planar graph. In the case of $\mathrm{DB}(6, 6, 0)$, to speed up the computation, we exploit Lemma 2.2 from [12].

Summarizing, we have the following

Lemma 6. *None of the dumbbells* $\mathrm{DB}(6, 6, l')$*,* $-3 \le l' \le 0$ *can be drawn as a thrackle.*

According to Lemma 6, Theorem 1 can be applied with $c = 6, l = 0$, and Theorem 2 follows.

For any $\varepsilon > 0$, our Theorem 1 and the above observations provide a deterministic algorithm with bounded running time to prove that all thrackles with n vertices have at most $(1 + \varepsilon)n$ edges or to exhibit a counterexample to Conway's conjecture.

In what follows, we estimate the dependence of the running time of our algorithm on ε. The analysis uses the standard random access machine model. In particular, we assume that all basic arithmetic operations can be carried out in constant time.

Theorem 7. *For any* $\varepsilon > 0$*, there is a deterministic algorithm with running time* $e^{O((1/\varepsilon^2)\ln(1/\varepsilon))}$ *to prove that all thrackles with* n *vertices have at most* $(1+\varepsilon)n$ *edges or to exhibit a counterexample to Conway's conjecture.*

Proof. First we estimate how long it takes for a given c and l, satisfying the assumptions in Theorem 1, to check whether there exists a dumbbell $\mathrm{DB}(c', c'', l')$ with c' and c'' even, $6 \le c' \le c'' \le c$, and with $-c'/2 \le l' \le l$, which can be drawn as a thrackle. Clearly, there are

$$\sum_{\substack{c'=6 \\ c' \text{ is even}}}^{c} \frac{(\frac{c'}{2} + l + 1)(c - c' + 2)}{2} = \frac{1}{8}lc^2 + \frac{1}{48}c^3 - \frac{3}{4}lc + l + \frac{1}{4}c^2 - \frac{25}{12}c + 3 \le \kappa(lc^2 + c^3)$$

dumbbells to check, for some $\kappa > 0$. In order to decide, whether a fixed dumbbell with m edges can be drawn as a thrackle, we construct at most $(m-2)!^m$ graphs, each with at most $O(m^2)$ edges, and we test each of them for planarity. Thus, the total running time of our algorithm is $O((lc^2+c^3)(2c+l-2)!^{2c+l}(2c+l)^2)$. Approximating the factorials by Stirling's formula, we can conclude that the running time is $O((2c+l)^{(2c+l)^2+\frac{1}{2}(2c+l)+5}e^{-(2c+l)})$.

Given $\epsilon > 0$, it can be shown by routine calculations that for $l \geq \frac{\kappa_l}{\varepsilon}$, for some κ_l, we can choose any c such that

$$c \geq \frac{\kappa l^2}{\varepsilon(l^2+12l+35)-(2l+14)}$$

for some $\kappa > 0$, so that the constant $\tau(c,l)$ introduced in Theorem 1 is at most $1+\varepsilon$. Thus, setting $c := \frac{\kappa_c}{\varepsilon}$ for some κ_c, Theorem 1 gives the required bound, i.e., at most $(1+\epsilon)n$. Plugging $\frac{\kappa_c}{\varepsilon}$ and $\frac{\kappa_l}{\varepsilon}$ as c and l, respectively, in $O((2c+l)^{(2c+l)^2+\frac{1}{2}(2c+l)+5}e^{-(2c+l)})$, the theorem follows. □

5 Concluding remarks

We say that two cycles C_1 and C_2 of a graph are at distance $l \geq 0$, if the length of a shortest path joining a vertex of C_1 to a vertex of C_2 is l. The following Turán-type questions were motivated by the proof of Theorem 1.

(1) Given two integers c_1, c_2, with $3 \leq c_1 \leq c_2$, what is the maximum number of edges that a planar graph on n vertices can have, if its girth is at least c_1, and no two cycles of length at most c_2 share an edge?

(2) Given three integers c_1, c_2, and l, with $3 \leq c_1 \leq c_2$ and $l \geq 0$, what is the maximum number of edges that a planar graph on n vertices can have, if its girth is at least c_1, and any two of its cycles of length at most c_2 are at distance larger than l ?

The inequalities (3) provide nontrivial upper bounds for restricted versions of the above problem for bipartite graphs.

We have the following general result.

Theorem 8. *Let c_1, c_2, and l denote three non-negative natural numbers with $3 \leq c_1 \leq c_2$. Let G be a planar graph with n vertices and girth at least c_1.*

(i) If no two cycles of length at most c_2 share an edge, then $|E(G)| \leq \frac{c_1c_2+c_1}{c_1c_2-c_2-1}n$.

(ii) If no two cycles of length at most c_2 are at distance at most l, then $|E(G)| \leq \frac{c_1c_2+2\lfloor l/2\rfloor c_2+2\lfloor l/2\rfloor+c_2+1}{2\lfloor l/2\rfloor c_2-2\lfloor l/2\rfloor+c_1c_2-c_1}n$.

Proof. (Outline.) Without loss of generality, we can assume in both cases that G is connected, it has no vertex of degree *one*, and it is not a cycle. To establish part (i), consider an embedding of G in the plane. Let $m = |E(G)|$, and let f and f_{c_2} stand for the number of faces of G and for the number of faces of length at most c_2. We follow the idea of the proof of Case (A), Theorem 1, with $f_{c_2} \leq \frac{1}{c_1}m$ instead of $f_c \leq \frac{1}{6}m$, and with the inequality

$$2m \geq c_1 f_{c_2} + (c_2+1)(f - f_{c_2})$$

replacing (1). Analogously, in the proof of part (ii), we use $f_{c_2} \leq \frac{1}{2\lfloor l/2\rfloor+c_1}n$ instead of the inequality $f_c \leq \frac{1}{2r+6}n$ □

It is possible that the constant factor in the part (i) of Theorem 8 is tight for all values of c_1 and c_2. It is certainly tight for all values of the form $c_1 = ml$ and $c_2 = m(l+1) - 1$, where m and l are natural numbers, as is shown by the following result, proof of which is omitted.

Theorem 9. *For any positive integers n_0, $m \geq 1$, and $l \geq 3$, one can construct a plane graph $G = (V, E)$ on at least n_0 vertices with girth ml such that all of its inner faces are of size ml or $m(l+1)$, its outer face is of size $2ml$, and each edge of G not on its outer face belongs to exactly one cycle of size ml, which is a face of G. The second smallest length of a cycle in G is $m(l+1)$.*

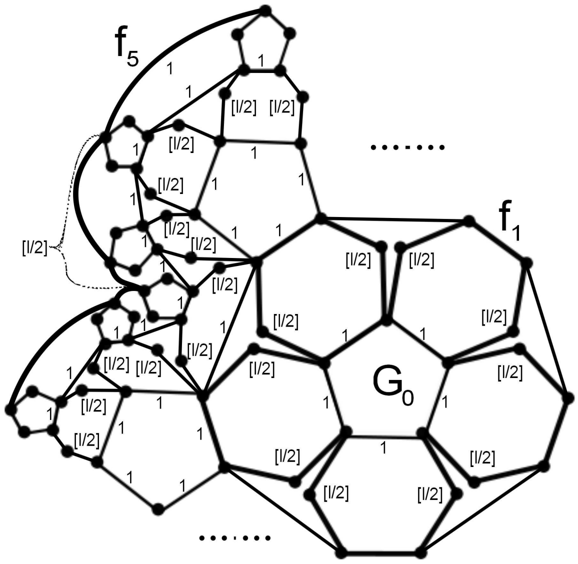

Figure 6: The key part of the construction from the proof of Theorem 9 for $l = 5$, and $m = 1$

.

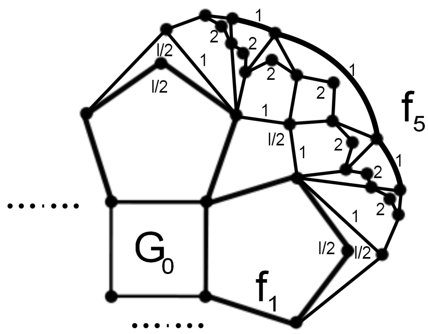

Figure 7: The key part of the construction from the proof of Theorem 9 for $l = 4$, and $m = 1$

.

Proof. Intuitively, one can think of a graph G meeting the requirements of the theorem as a “generalized chessboard” with white and black fields (faces) of size ml and $m(l+1)$, respectively.

Observe that it is enough to provide a construction for $m = 1$. Indeed, given a construction G for some $l = l'$, $n_0 = n'_0$, and $m = 1$, for any $m' > 1$, one can subdivide each edge of G into m' pieces to obtain a valid construction for $l := l'$, $n_0 := n'_0$, and $m := m'$.

Here we consider only the case when l is *odd*; the other case can be treated analogously. We construct G recursively, starting from a plane graph G_0, that is a cycle of length l, as depicted in Figure 6. Let f_i denote the outer face of G_i, $i = 0, 1, 2, \ldots$ (for $i > 1$, the outer faces f_i are not completely depicted in the figure). Our construction satisfies the condition that each edge of G lies exactly on one outerface f_i for some i.

For any $i \geq 0$, we obtain G_{2i+1} from G_{2i}, by attaching faces of size $l + 1$ along f_{2i} in the way indicated in Figure 6 (the labels of the paths in the Figure indicate the length). Analogously, for any $i \geq 1$, the graph G_{2i} can be obtained from G_{2i-1}, by attaching to the sides of f_{2i-1} faces of size l, in the way indicated in the figure. Observe that Figure 6 can be easily modified to work for any odd value of l, and it is not hard to obtain similar construction for even values either (see Figure 7). The key feature of the construction is that the length of the outer face f_1 is the same as the length of the outer face f_5 (both are drawn with thicker lines in the figure). Thus, we can repeat the pattern consisting of the outer faces $f_1, \ldots, f_5$ until the number of vertices in G is at least n_0, and then finish with a graph G_{4i+2}, for some i. Notice, that during this process we never create multiple edges, and that each edge lies on the outer face of exactly one G_i.

In what follows, we show that the girth of G is l and that no two cycles of length l share an edge, which concludes the proof. To this end we show that a smallest cycle C in G is a face cycle of length l. In order to see that one can proceed by distinguishing, whether there are vertices of C belonging to the outer face f_{4i+1}, and vertices belonging to the outer face f_{4i+5}, for some i, or there are not such vertices. If that is the case, the length of C is bigger than l. Indeed, a shortest path between f_{4i+1} and f_{4i+5} is of length $l - 1$. Now, we can proceed by checking a small subgraph of G. □

If we slightly relax the conditions in Theorem 8 by forbidding only dumbbells determined by *face cycles*, we obtain some tight bounds. For instance, it is not hard to prove the following.

Theorem 10. *Let c_1 and c_2 be two nonnegative integers with $3 \leq c_1 \leq c_2$. Let G be a plane graph on n vertices, which has no face shorter than c_1 and no two faces of length at most c_2 that share an edge. Then we have*

$$|E(G)| \leq \frac{c_1c_2 + c_1}{c_1c_2 - c_2 - 1}n,$$

and the inequality does not remain true with any smaller constant.

Proof. The proof of the upper bounds is the same as the proof of Theorem 8. In order to show the tightness we construct a plane graph G whose inner faces have size c_1 or $c_2 + 1$, and whose outer face has size bounded from above by a function depending only on c_2. Moreover, no two faces in G of size c_1 share an edge and each edge belongs to exactly one face of size c_1. Clearly, G contains as many edges as the stated upper bound up to an additive constant depending only on c_1 and c_2.

Again, we construct G in an inductive way as in the proof of Theorem 9 starting from a plane graph G_0, that is a face cycle of length c_1. Let f_i denote the outer face of G_i, $i \in \mathbb{N}$. We construct G_{2i+1} from G_{2i}, $i \in \mathbb{N}$, by attaching faces of size $c_2 + 1$ along f_{2i}. Analogously, we construct G_{2i} from G_{2i-1}, $i > 0$, by attaching faces of size c_1 along f_{2i-1}. We have quite freedom in the way, how we attach the new faces. However, we require that faces attached during a single step are edge-wise pairwise disjoint, and that each edge lies on the outer face in exactly one G_i for some $i \in \mathbb{N}$. We also want to avoid creating multiple edges.

Clearly, we can keep increasing the number of vertices of G_i without introducing multiple edges with i tending to infinity. Moreover, whenever $|f_i|$ is bigger than $2(c_2+1)$, we can also decrease in G_{i+2} the size of the outer face f_{i+2} with respect to f_i, so that $|f_{i+2}|$ is still of size at least (c_2+1). To this end one can proceed in many ways, out of which we choose the following one.

If i is odd (if i is even and $c_1 > 3$ we can proceed in the same fashion) and $|f_i| > 2(c_2+1)$, let $l' = \lfloor |f_i|/(c_2+1) \rfloor$. Let $v_0, \ldots v_{|f_i|-1}$ denote the vertices on f_i indexed in correspondence with their clockwise order around f_i. We divide f_i into $l'-1$ paths $P_j = v_{j(c_2+1)} \ldots v_{(j+1)(c_2+1)}$, for $j = 0 \ldots l'-3$, and $P_{l'-2} = v_{(l'-2)(c_2+1)} \ldots v_0$. Then along each P_j, for $j = 0 \ldots l'-3$, we attach two faces of size l whose intersections with f_i are the paths of length $\lfloor (c_2+1)/2 \rfloor$ and $\lceil (c_2+1)/2 \rceil$, respectively. Notice that the length l_p of $P_{l'-2}$ is at least $2(c_2+1)$ and at most $3(c_2+1)-1$. Thus, we can either attach three faces of size c_2+1 along $P_{l'-2}$, if $l_p = 2(c_2+1)$, or four faces of size (c_2+1) otherwise. Note that we can be forced to create multiple edges if $c_2+1 < 7$. However, since $c_2+1 \geq 4$, it can happen only when we attach a face along $P_{l'-2}$. Observe that whenever we have an edge $v_p v_r$ not lying on f_i, such that v_p and v_r belongs to f_i, $v_{p+1}v_{r+1} \notin E(G_i)$ (indices are taken modulo $|f_i|$). Using that observation it is a simple case analysis to show that by shifting the labels v_i-s on the outer face by one position clockwise or counter clockwise, we can avoid creating a multiple edge.

If i is even, $c_1 = 3$, and we proceed as above, we can create many multiple edges, but at most one along each P_j, $0 \leq j \leq l'-3$. However, using the above observation we can see that one of two ways, how we can attach two faces of length 3 along such P_j, does not create a multiple edge. Since in the beginning we required, that we do not have to be able to decrease the number of edges on the outer face at any single step, but after two steps, it is enough to show that we can attach the faces along $P_{l'-2}$ so that we have $|f_{i+1}| \leq |f_i|$. We again proceed by a simple case analysis using the same observation as above.

Notice, that each part of f_{i+1} that was attached along P_j, for $j = 0 \ldots l'-3$, has the same length as P_j, and that the part of f_{i+1} that was attached along $P_{l'-2}$ has the length strictly smaller than $P_{l'-2}$, if i is odd. Clearly, we can keep the length of f_{i+1} always at least c_2+1. Thus, for any n_0 we can achieve that some G_z has at least n_0 vertices and the length of its outer face is at most $2(c_2+1)$ and at least c_2+1. □

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Appendix

A Backtracking algorithm

For sake of completeness in this section we describe a backtracking algorithm checking, whether a given dumbbell $G = (V, E)$ can be drawn as a thrackle. We orient the edges of G, so that we can traverse them by a single walk, so called Euler's walk, during which we visit each edge just once. We use the notation from Section 4.

Let us start with a description of the routines used by our algorithm.

The routine UPDATE(π_e, e', pos) returns the updated permutation $\pi_e \in {E'}_e^{m'(e)}$, which corresponds to adding one more crossing vertex to an already constructed part of (G', Π') corresponding to a subgraph $G' = (V', E')$ of G, where $e \in E$, $e' \in E'$, $m'(e)$ returns the number of crossings of e already modeled by (G', Π'), and $\Pi'(G') := \{\pi_e \in {E'}_e^{m'(e)} | e \in E'\}$. UPDATE($\pi(e)$, e', pos) returns the permutation π'_e whose length is by one longer than π_e, such that

$$\pi'_e(i) = \begin{cases} \pi_e(i) & \text{if } i < pos \\ e' & \text{if } i = pos \\ \pi_e(i-1) & \text{if } i > pos \end{cases}$$

REVERSE_UPDATE(π_e, e', pos) corresponds to the reverse operation of the operation UPDATE(π_e, e', pos). PICK_NEXT_EDGE(G) returns a next edge in our Euler's walk. In order to check, whether G can be drawn as a thrackle the algorithm just calls the procedure BACKTRACKING(e) for an edge $e \in E$. The algorithm returns *true* if G can be drawn as a thrackle, and it returns *false* if G cannot be drawn as a thrackle. In our description of the algorithm we restrain from all optimization details, which were mentioned in Section 4. The pseudocode of the backtracking routine follows.

Algorithm 1: Thrackleabilty testing

```
BACKTRACKING (e ∈ E(G))
begin
    if (G′, Π′) cannot be extended then
        return true
    if e = -1 then
        e = PICK_NEXT_EDGE(G)
    if e has crossed all edges in E′_e then
        BACKTRACKING(-1)
    else
        forall e′ ∈ E′_e which e has not already crossed do
            for pos = 1 to length(π_e′) do
                π_e′ = UPDATE(π_e′, e, pos)
                π_e = UPDATE(π_e, e′, LENGTH(π_e)+1)
                if IS_PLANAR((G′, Π′)) then
                    if BACKTRACKING(e) then
                        return true
                    else
                        REVERSE_UPDATE(π_e′, e, pos)
                        REVERSE_UPDATE(π_e, e′, LENGTH(π_e))
                else
                    REVERSE_UPDATE(π_e′, e, pos)
                    REVERSE_UPDATE(π_e, e′, LENGTH(π_e))
            end
        end
    return false
end
```