The number of crossings in multigraphs with no empty lens

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Abstract. Let G be a multigraph with n vertices and $e > 4n$ edges, drawn in the plane such that any two parallel edges form a simple closed curve with at least one vertex in its interior. Pach and Tóth [5] extended the Crossing Lemma of Ajtai *et al.* [1] and Leighton [3] by showing that if no two adjacent edges cross and every pair of nonadjacent edges cross at most once, then the number of edge crossings in G is at least $\alpha e^3/n^2$, for a suitable constant $\alpha > 0$. The situation turns out to be quite different if nonparallel edges are allowed to cross any number of times. It is proved that in this case the number of crossings in G is at least $\alpha e^{2.5}/n^{1.5}$. The order of magnitude of this bound cannot be improved.

1 Introduction

In this paper, multigraphs may have parallel edges but no loops. A topological graph (or multigraph) is a graph (multigraph) G drawn in the plane with the property that every vertex is represented by a point and every edge uv is represented by a curve (continuous arc) connecting the two points corresponding to the vertices u and v. We assume, for simplicity, that the points and curves are in "general position", that is, (a) no edge passes through any vertex different from its endpoints; (b) any pair of edges intersect in at most finitely many points; (c) if two edges share an interior point, then they properly cross at this point; and (d) no 3 edges cross at the same point. Throughout this paper, every multigraph G is a topological multigraph, that is, G is considered with a fixed drawing that is given from the context. In notation and terminology, we then do not distinguish between the vertices (edges) and the points (curves) representing them. The number of crossing points in the considered drawing of G is called its *crossing number*, denoted by $\mathrm{cr}(G)$. (I.e., $\mathrm{cr}(G)$ is defined for topological multigraphs rather than abstract multigraphs.)

The classic "crossing lemma" of Ajtai, Chvátal, Newborn, Szemerédi [1] and Leighton [3] gives an asymptotically best-possible lower bound on the crossing number in any n-vertex e-edge topological graph without loops or parallel edges, provided $e > 4n$.

Theorem A (Crossing Lemma, Ajtai *et al.* [1] and Leighton [3]) *There is an absolute constant $\alpha > 0$, such that for any n-vertex e-edge topological graph G we have*

$$\mathrm{cr}(G) \geq \alpha \frac{e^3}{n^2}, \qquad \textit{provided } e > 4n.$$

In general, the Crossing Lemma does not hold for topological multigraphs with parallel edges, as for every n and e there are n-vertex e-edge topological multigraphs G with $\mathrm{cr}(G) = 0$. Székely proved the following variant for multigraphs by restricting the edge multiplicity, that is the maximum number of pairwise parallel edges, in G to be at most m.

Theorem B (Székely [6]) *There is an absolute constant $\alpha > 0$ such that for any $m \geq 1$ and any n-vertex e-edge multigraph G with edge multiplicity at most m we have*

$$\mathrm{cr}(G) \geq \alpha \frac{e^3}{mn^2}, \qquad \textit{provided } e > 4mn.$$

Most recently, Pach and Tóth [5] extended the Crossing Lemma to so-called branching multigraphs. We say that a topological multigraph is

- *separated* if any pair of parallel edges form a simple closed curve with at least one vertex in its interior and at least one vertex in its exterior,
- *single-crossing* if any pair of edges cross at most once (that is, edges sharing k endpoints, $k \in \{0, 1, 2\}$, may have at most $k + 1$ points in common), and
- *locally starlike* if no two adjacent edges cross (that is, edges sharing k endpoints, $k \in \{1, 2\}$, may not cross).

A topological multigraph is *branching* if it is separated, single-crossing and locally starlike. Note that the edge multiplicity of a branching multigraph may be as high as $n - 2$.

Theorem C (Pach and Tóth [5]) *There is an absolute constant $\alpha > 0$ such that for any n-vertex e-edge branching multigraph G we have*

$$\mathrm{cr}(G) \geq \alpha \frac{e^3}{n^2}, \qquad \textit{provided } e > 4n.$$

In this paper we generalize Theorem C by showing that the Crossing Lemma holds for all topological multigraphs that are separated and locally starlike, but not necessarily single-crossing. We also prove a Crossing Lemma variant for separated (and not necessarily locally starlike) multigraphs, where however the term $\alpha \frac{e^3}{n^2}$ must be replaced by $\alpha \frac{e^{2.5}}{n^{1.5}}$. Both results are best-possible up to the value of constant α.

Theorem 1. *There is an absolute constant $\alpha > 0$ such that for any n-vertex e-edge topological multigraph G with $e > 4n$ we have*

(i) $\mathrm{cr}(G) \geq \alpha \frac{e^3}{n^2}$, *if G is separated and locally starlike.*

(ii) $\mathrm{cr}(G) \geq \alpha \frac{e^{2.5}}{n^{1.5}}$*, if* G *is separated.*

Moreover, both bounds are best-possible up to the constant α*.*

We prove Theorem 1 in Section 3. Our arguments hold in a more general setting, which we present in Section 2. In Section 4 we use this general setting to deduce other known Crossing Lemma variants, including Theorem B. We conclude the paper with some open questions in Section 5.

2 A Generalized Crossing Lemma

In this section we consider general drawing styles and propose a generalized Crossing Lemma, which will subsume all Crossing Lemma variants mentioned here. A *drawing style* D is a predicate over the collection of all topological drawings, i.e., for each topological drawing of a multigraph G we specify whether G is in drawing style D or not. We say that G is a multigraph in drawing style D when G is a topological multigraph whose drawing is in drawing style D.

In order to prove our generalized Crossing Lemma, we follow the line of arguments of Pach and Tóth [5] for branching multigraphs. Their main tool is a bisection theorem for branching drawings, which easily generalizes to all separated drawings. We generalize their definition as follows.

Definition 1 (D-bisection width). *For a drawing style* D *the* D-bisection width $\mathrm{b}_D(G)$ *of a multigraph* G *in drawing style* D *is the smallest number of edges whose removal splits* G *into two multigraphs,* G_1 *and* G_2*, in drawing style* D *with no edge connecting them such that* $|V(G_1)|, |V(G_2)| \geq n/5$*.*

Given a topological multigraph G, we call any operation of the following form a *vertex split*: (1) Replace a vertex v of G by two vertices v_1 and v_2 and (2) by locally modifying the edges in a small neighborhood of v, connect each edge in G incident to v to either v_1 or v_2 in such a way that no new crossing is created.

We say that a drawing style is *monotone* and *split-compatible* if removing edges and performing vertex splits both retains the drawing style, that is, D is monotone and split-compatible if for every multigraph G in drawing style D for any edge removal, as well as, any vertex split, the resulting multigraph with its inherited drawing from G is again in drawing style D.

Note that we require a monotone drawing style to be retained only while removing edges, but not necessarily while removing vertices. For example, the branching drawing style is in general not maintained after removing a vertex, since a closed curve formed by a pair of parallel edges might become empty.

We are now ready to state our main result.

Theorem 2 (generalized Crossing Lemma). *Suppose* D *is a monotone and split-compatible drawing style, and that there are constants* $k_1, k_2, k_3 > 0$ *and* $b > 1$ *such that each of the following holds for every* n'*-vertex* e'*-edge multigraph* G' *in drawing style* D*:*

(P1) *If* $\mathrm{cr}(G') = 0$*, then the edge count satisfies* $e' \le k_1 \cdot n'$*.*
(P2) *The* D*-bisection width satisfies* $b_D(G') \le k_2\sqrt{\mathrm{cr}(G') + \Delta(G') \cdot e' + n'}$*.*
(P3) *The edge count satisfies* $e' \le k_3 n'^b$*.*

Then there exists an absolute constant $\alpha > 0$ *such that for any* n*-vertex* e*-edge multigraph* G *in drawing style* D *we have*

$$\mathrm{cr}(G) \ge \alpha \frac{e^{x(b)+2}}{n^{x(b)+1}}, \qquad \textit{provided } e > (k_1 + 1)n,$$

where $x(b) := 1/(b-1)$ *and* $\alpha = \alpha_b \cdot k_2^{-2} \cdot k_3^{-x(b)}$ *for some constant* α_b *depending only on* b*.*

Lemma 1. *If there exist for arbitrarily large* n *multigraphs in drawing style* D *with* n *vertices and* $e = \Theta(n^b)$ *edges such that any two edges cross at most a constant number of times, then the bound in Theorem 2 is asymptotically tight.*

Proof. Consider such an n-vertex e-edge multigraph in drawing style D. Clearly, there are at most $O(e^2) = O(n^{2b})$ crossings, while Theorem 2 gives with $x(b) = 1/(b-1)$ that there are at least

$$\Omega\left(\frac{e^{x(b)+2}}{n^{x(b)+1}}\right) = \Omega\left(\frac{e^{x(b)+2}}{n^{b \cdot x(b)}}\right) = \Omega\left(\frac{n^{b \cdot x(b)+2b}}{n^{b \cdot x(b)}}\right) = \Omega\left(n^{2b}\right)$$

crossings.

2.1 Proof of Theorem 2

We define an absolute constant

$$\alpha := \frac{1}{2^{2x(b)+14}} \cdot \frac{1}{k_2^2} \cdot \frac{1}{k_3^{x(b)}} \tag{1}$$

Now let $\tilde{G}$ be a fixed multigraph in drawing style D with $\tilde{n}$ vertices and $\tilde{e} > (k_1 + 1)\tilde{n}$ edges. Let G' be an edge-maximal subgraph of $\tilde{G}$ on vertex set $V(\tilde{G})$ such that the inherited drawing of G' has no crossings. Since D is monotone, G' is in drawing style D. Hence, by **(P1)**, for the number e' of edges in G' we have $e' \le k_1 \cdot n' = k_1 \cdot \tilde{n}$. Since G' is edge-maximal crossing-free, each edge in $E(\tilde{G}) - E(G')$ has at least one crossing with an edge in $E(G')$. Thus

$$\mathrm{cr}(\tilde{G}) \ge \tilde{e} - e' \ge \tilde{e} - k_1\tilde{n} > \tilde{n}. \tag{2}$$

In case $\tilde{e} \le \delta\tilde{n}$ for $\delta := \alpha^{-1/(x(b)+2)}$, we get

$$\mathrm{cr}(\tilde{G}) \overset{(2)}{>} \tilde{n} \ge \alpha \cdot \frac{\tilde{e}^{x(b)+2}}{\tilde{n}^{x(b)+1}},$$

as desired. To prove Theorem 2, suppose for the sake of contradiction, that $\tilde{e} > \delta\tilde{n}$ and that the number of crossings in $\tilde{G}$ satisfies

$$\mathrm{cr}(\tilde{G}) < \alpha \cdot \frac{\tilde{e}^{x(b)+2}}{\tilde{n}^{x(b)+1}}.$$

Let d denote the average degree of the vertices of $\tilde{G}$, that is, $d = 2\tilde{e}/\tilde{n}$. For every vertex $v \in V(\tilde{G})$ whose degree, $\deg(v, \tilde{G})$, is larger than d, we perform $\lceil \deg(v, \tilde{G})/d \rceil - 1$ vertex splits so as to split v into $\lceil \deg(v, \tilde{G})/d \rceil$ vertices, each of degree at most d. At the end of the procedure, we obtain a multigraph G with $e = \tilde{e}$ edges, $n < 2\tilde{n}$ vertices, and maximum degree $\Delta(G) \le d = 2\tilde{e}/\tilde{n} < 4e/n$. Moreover, as D is split-compatible, G is in drawing style D. For the number of crossings in G, we have

$$\mathrm{cr}(G) = \mathrm{cr}(G') < \alpha \cdot \frac{\tilde{e}^{x(b)+2}}{\tilde{n}^{x(b)+1}} < 2^{x(b)+1}\alpha \cdot \frac{e^{x(b)+2}}{n^{x(b)+1}}. \tag{3}$$

Moreover, recall that

$$e > \delta\tilde{n} > \delta\frac{n}{2} \qquad \text{for } \delta = \frac{1}{\alpha^{1/(x(b)+2)}}. \tag{4}$$

We break G into smaller parts, according to the following procedure. At each step the parts form a partition of the entire vertex set $V(G)$.

Decomposition Algorithm

Step 0.
▷ **Let** $G^0 = G, G_1^0 = G, M_0 = 1, m_0 = 1$.

Suppose that we have already executed Step i, and that the resulting graph in drawing style D, G^i, consists of M_i parts, $G_1^i, G_2^i, \ldots, G_{M_i}^i$, each having at most $(4/5)^i n$ vertices. Assume without loss of generality that the first m_i parts of G^i have at least $(4/5)^{i+1}n$ vertices and the remaining $M_i - m_i$ have fewer. Letting $n(G_j^i)$ denote the number of vertices of the part G_j^i, we have

$$(4/5)^{i+1}n(G) \le n(G_j^i) \le (4/5)^i n(G), \qquad 1 \le j \le m_i. \tag{5}$$

Hence,

$$m_i \le (5/4)^{i+1}. \tag{6}$$

Step $i+1$.
▷ **If**

$$(4/5)^i < \frac{1}{(2k_3)^{x(b)}} \cdot \frac{e^{x(b)}}{n^{x(b)+1}}, \tag{7}$$

then stop.
▷ **Else**, for $j = 1, 2, \ldots, m_i$, delete $\mathrm{b_D}(G_j^i)$ edges from G_j^i, as guaranteed by **(P2)**, such that G_j^i falls into two parts, each of which is in drawing style D and contains at most $(4/5)n(G_j^i)$ vertices. Let G^{i+1} denote the resulting graph on the original set of n vertices.

Clearly, each part of G^{i+1} has at most $(4/5)^{i+1}n$ vertices.

Suppose that the DECOMPOSITION ALGORITHM terminates in STEP $k+1$. If $k > 0$, then

$$(4/5)^k < \frac{1}{(2k_3)^{x(b)}} \cdot \frac{e^{x(b)}}{n^{x(b)+1}} \le (4/5)^{k-1}. \tag{8}$$

First, we give an upper bound on the total number of edges deleted from G. Using the fact that, for any nonnegative numbers $a_1, \ldots, a_m$,

$$\sum_{j=1}^{m} \sqrt{a_j} \le \sqrt{m \sum_{j=1}^{m} a_j}, \tag{9}$$

we obtain that, for any $0 \le i \le k$,

$$\begin{aligned} \sum_{j=1}^{m_i} \sqrt{\mathrm{cr}(G_j^i)} &\overset{(9)}{\le} \sqrt{m_i \sum_{j=1}^{m_i} \mathrm{cr}(G_j^i)} \overset{(6)}{\le} \sqrt{(5/4)^{i+1}}\sqrt{\mathrm{cr}(G)} \\ &\overset{(3)}{<} \sqrt{(5/4)^{i+1}}\sqrt{2^{x(b)+1}\alpha \cdot \frac{e^{x(b)+2}}{n^{x(b)+1}}}. \end{aligned} \tag{10}$$

Letting $e(G_j^i)$ and $\Delta(G_j^i)$ denote the number of edges and maximum degree in part G_j^i, respectively, we obtain similarly

$$\begin{aligned} \sum_{j=1}^{m_i} \sqrt{\Delta(G_j^i) \cdot e(G_j^i) + n(G_j^i)} &\overset{(9)}{\le} \sqrt{m_i \left(\sum_{j=1}^{m_i} \Delta(G_j^i) \cdot e(G_j^i) + n(G_j^i) \right)} \\ &\overset{(6)}{\le} \sqrt{(5/4)^{i+1}}\sqrt{\Delta(G) \cdot e + n} \le \sqrt{(5/4)^{i+1}}\sqrt{\frac{4e}{n}e + n} \\ &< \sqrt{(5/4)^{i+1}}\sqrt{\frac{5e^2}{n}} < \sqrt{(5/4)^{i+1}}\frac{3e}{\sqrt{n}}, \end{aligned} \tag{11}$$

where we used in the last line the fact that $n < e$.

Using a partial sum of a geometric series we get

$$\sum_{i=0}^{k} (\sqrt{5/4})^{i+1} = \frac{(\sqrt{5/4})^{k+2} - 1}{\sqrt{5/4} - 1} - 1 < \frac{(\sqrt{5/4})^3}{\sqrt{5/4} - 1} \cdot (\sqrt{5/4})^{k-1} < 12 \cdot (\sqrt{5/4})^{k-1} \tag{12}$$

Thus, by **(P2)**, the total number of edges deleted during the decomposition procedure is

$$\begin{aligned} \sum_{i=0}^{k} \sum_{j=1}^{m_i} \mathrm{b}_D(G_j^i) &\le k_2 \sum_{i=0}^{k} \sum_{j=1}^{m_i} \sqrt{\mathrm{cr}(G_j^i) + \Delta(G_j^i) \cdot e(G_j^i) + n(G_j^i)} \\ &\le k_2 \left(\sum_{i=0}^{k} \sum_{j=1}^{m_i} \sqrt{\mathrm{cr}(G_j^i)} + \sum_{i=0}^{k} \sum_{j=1}^{m_i} \sqrt{\Delta(G_j^i) \cdot e(G_j^i) + n(G_j^i)} \right) \end{aligned}$$

$$
\begin{aligned}
&\overset{(10),(11)}{\le} k_2 \left(\sum_{i=0}^{k} \sqrt{(5/4)^{i+1}}\right)\left(\sqrt{2^{x(b)+1}\alpha \cdot \frac{e^{x(b)+2}}{n^{x(b)+1}}} + \frac{3e}{\sqrt{n}}\right)\\
&\overset{(12)}{<} k_2 \cdot 12\sqrt{(5/4)^{k-1}}\left(\sqrt{2^{x(b)+1}\alpha \cdot \frac{e^{x(b)+2}}{n^{x(b)+1}}} + \frac{3e}{\sqrt{n}}\right)\\
&\overset{(8)}{<} k_2 \cdot 12\sqrt{(2k_3)^{x(b)} \cdot \frac{n^{x(b)+1}}{e^{x(b)}}}\left(\sqrt{2^{x(b)+1}\alpha \cdot \frac{e^{x(b)+2}}{n^{x(b)+1}}} + \frac{3e}{\sqrt{n}}\right)\\
&< k_2 \cdot 36 \cdot \sqrt{k_3^{x(b)}}\left(2^{x(b)}\sqrt{\alpha}e + \sqrt{\frac{2^{x(b)}n^{x(b)}}{e^{x(b)-2}}}\right)\\
&\overset{(4)}{<} k_2 \cdot 36 \cdot \sqrt{k_3^{x(b)}}\left(2^{x(b)}\sqrt{\alpha} + \sqrt{\frac{1}{\delta^{x(b)}}}\right)e\\
&\overset{(4)}{=} k_2 \cdot 36 \cdot \sqrt{k_3^{x(b)}}\left(2^{x(b)}\sqrt{\alpha} + \sqrt{\alpha^{\frac{x(b)}{x(b)+2}}}\right)e < k_2 \cdot \sqrt{k_3^{x(b)}} \cdot 2^{x(b)+6}\sqrt{\alpha}e \overset{(1)}{=} \frac{e}{2}.
\end{aligned}
\tag{13}
$$

By (13) the Decomposition Algorithm removes less than half of the edges of G if $k > 0$. Hence, the number of edges of the graph G^k obtained in the final step of this procedure satisfies

$$
e(G^k) > \frac{e}{2}. \tag{14}
$$

(Note that this inequality trivially holds if the algorithm terminates in the very first step, i.e., when $k = 0$.)

Next we shall give an upper bound on $e(G^k)$ that contradicts (14). The number of vertices of each part G_j^k of G^k satisfies

$$
n(G_j^k) \le (4/5)^k n \overset{(8)}{<} \left(\frac{1}{(2k_3)^{x(b)}} \cdot \frac{e^{x(b)}}{n^{x(b)+1}}\right) n = \left(\frac{e}{2 \cdot k_3 \cdot n}\right)^{x(b)}, \quad 1 \le j \le M_k.
$$

Hence

$$
n(G_j^k)^{b-1} < \left(\frac{e}{2 \cdot k_3 \cdot n}\right)^{x(b)(b-1)} = \frac{e}{2 \cdot k_3 \cdot n},
$$

since $x(b) = 1/(b-1)$ and hence $x(b)(b-1) = 1$.

By **(P3)**, we have

$$
e(G_j^k) \le k_3 \cdot n(G_j^k)^b < k_3 \cdot n(G_j^k) \cdot \frac{e}{2 \cdot k_3 \cdot n} = n(G_j^k) \cdot \frac{e}{2n}.
$$

Therefore, for the total number of edges of G^k we have

$$
e(G^k) = \sum_{j=1}^{M_k} e(G_j^k) < \frac{e}{2n}\sum_{j=1}^{M_k} n(G_j^k) = \frac{e}{2},
$$

contradicting (14). This completes the proof of Theorem 2. $\square$

3 Separated Multigraphs

We derive our Crossing Lemma variants for separated multigraphs (Theorem 1) from the generalized Crossing Lemma (Theorem 2) presented in Section 2. Let us denote the separated drawing style by D_{sep} and the separated and locally starlike drawing style by $D_{\text{loc-star}}$. In order to apply Theorem 2, we shall find for $D = D_{\text{sep}}, D_{\text{loc-star}}$ **(1)** the largest number of edges in a crossing-free n-vertex multigraph in drawing style D, **(2)** an upper bound on the D-bisection width of multigraphs in drawing style D, and **(3)** an upper bound on the number of edges in any n-vertex multigraph in drawing style D.

As for crossing-free multigraphs D_{sep} and $D_{\text{loc-star}}$ are equivalent to the branching drawing style, we can rely on the the following Lemma of Pach and Tóth.

Lemma 2 (Pach and Tóth [5]). *Any n-vertex crossing-free branching multigraph, $n \geq 3$, has at most $3n-6$ edges.*

Corollary 1. *Any n-vertex crossing-free multigraph in drawing style D_{sep} or $D_{\text{loc-star}}$, $n \geq 3$, has at most $3n-6$ edges.*

Also we can derive the bounds on the D-bisection width from the corresponding bound for the branching drawing style due to Pach and Tóth.

Lemma 3 (Pach and Tóth [5]). *For any multigraph G in the branching drawing style D with n vertices of degrees $d_1, d_2, \ldots, d_n$, and with $\operatorname{cr}(G)$ crossings, the D-bisection width of G satisfies*

$$\mathrm{b}_D(G) \leq 22\sqrt{\operatorname{cr}(G) + \sum_{i=1}^{n} d_i^2 + n}.$$

Lemma 4. *For $D = D_{\text{sep}}, D_{\text{loc-star}}$ any multigraph G in the drawing style D with n vertices, e edges, maximum degree $\Delta(G)$, and with $\operatorname{cr}(G)$ crossings, the D-bisection width of G satisfies*

$$\mathrm{b}_D(G) \leq 44\sqrt{\operatorname{cr}(G) + \Delta(G) \cdot e + n}.$$

Proof. Let G be a multigraph in drawing style D. Suppose there is a simple closed curve γ formed by parts of only two edges e_1 and e_2, which does not have a vertex in its interior or no vertex in its exterior. This can happen between two consecutive crossings of e_1 and e_2, or for $D \neq D_{\text{loc-star}}$ between a common endpoint and a crossing of e_1 and e_2. Say e_1 has at most as many crossings along γ as e_2. We then reroute the part of e_2 on γ very closely along the part of e_1 along γ so as to reduce the number of crossings between e_1 and e_2. Note that the resulting multigraph is again in drawing style D and has at most as many crossings as G.

Hence, we can redraw G to obtain a multigraph G' in drawing style D with $\operatorname{cr}(G') \leq \operatorname{cr}(G)$, such that introducing a new vertex at each crossing of G' creates a crossing-free multigraph that is separated, i.e., in drawing style D. Now, using precisely the same proof as the proof of its special case Lemma 3 in [5], we can show that

$$\mathrm{b}_D(G') \leq 22\sqrt{\mathrm{cr}(G') + \sum_{i=1}^{n} d_i^2 + n},$$

where $d_1, \ldots, d_n$ denote the degrees of vertices in G'. Thus with

$$\sum_{i=1}^{n} d_i^2 \leq \Delta(G) \sum_{i=1}^{n} d_i \leq 2\Delta(G) \cdot e$$

the result follows.

Finally, let us bound the number of edges in crossing-free multigraphs. Again, we can reuse the result of Pach and Tóth for the branching drawing style.

Lemma 5 (Pach and Tóth [5]). *For any n-vertex e-edge, $n \geq 3$, multigraph of maximum degree $\Delta(G)$ in the branching drawing style we have $\Delta(G) \leq 2n-4$ and $e \leq n(n-2)$, and both bounds are best-possible.*

Lemma 6. *For any n-vertex e-edge multigraph in drawing style D of maximum degree $\Delta(G)$ we have*

(i) $\Delta(G) \leq (n-1)(n-2)$ and $e \leq \binom{n}{2}(n-2)$ if $D = D_{\mathrm{sep}}$,
(ii) $\Delta(G) \leq 2n-4$ and $e \leq n(n-2)$ if G if $D = D_{\mathrm{loc-star}}$.

Moreover, each bound is best-possible.

Proof. Let G be a fixed n-vertex, $n \geq 3$, e-edge crossing-free multigraph in drawing style D.

(i) Let $D = D_{\mathrm{sep}}$. Clearly, every set of parallel edges contains at most $n-2$ edges, since every lens has to contain a vertex different from the two endpoints of these edges. This gives $\Delta(G) \leq (n-1)(n-2)$ and $e \leq n\Delta(G)/2 = \binom{n}{2}(n-2)$. To see that these bounds are tight, consider n points in the plane with no four points on a circle. Then it is easy to draw between any two points $n-2$ edges as circular arcs such that the resulting multigraph (which has $\binom{n}{2}(n-2)$ edges) is in separating drawing style.
(ii) Let $D = D_{\mathrm{loc-star}}$. Consider any fixed vertex v in G and remove all edges not incident to v. The resulting multigraph is branching and hence by Lemma 5 v has at most $2n-4$ incident edges. Thus $\Delta(G) \leq 2n-4$ and $e \leq n\Delta(G)/2 = n(n-2)$. By Lemma 5, these bounds are tight, even for the more restrictive branching drawing style.

Hence, by Corollary 1 and Lemmas 4 and 6,

We are now ready to prove that drawing styles $D_{\mathrm{loc-star}}$ and D_{sep} fulfill the requirements of the generalized Crossing Lemma (Theorem 2), which lets us prove Theorem 1.

Proof (Proof of Theorem 1). Let $D = D_{\mathrm{loc-star}}$ for (i) and $D = D_{\mathrm{sep}}$ for (ii). Clearly, these drawing styles are monotone, i.e., maintained when removing edges, as well as split-compatible. So it remains to determine the constants $k_1, k_2, k_3 > 0$ and $b > 1$ such that **(P1)**, **(P2)**, and **(P3)** hold for D.

(P1) holds with $k_1 = 3$ for $D = D_{\mathrm{loc-star}}, D_{\mathrm{sep}}$ by Corollary 1. **(P2)** holds with $k_2 = 44$ for $D = D_{\mathrm{sep}}$ by Lemma 4, which implies the same for $D = D_{\mathrm{loc-star}}$. **(P3)** holds with $k_3 = 1$ and $b = 3$ for $D = D_{\mathrm{sep}}$ by Lemma 6(i), and with $k_3 = 1$ and $b = 2$ for $D = D_{\mathrm{loc-star}}$ by Lemma 6(ii).

For $b = 2$ we have $x(b) = 1/(b-1) = 1$. Thus Theorem 2 for $D = D_{\mathrm{loc-star}}$ gives an absolute constant $\alpha > 0$ such that for every n-vertex e-edge separated and locally starlike multigraph we have $\mathrm{cr}(G) \geq \alpha e^{x(b)+2}/n^{x(b)+1} = \alpha e^3/n^2$, provided $e > (k_1 + 1)n = 4n$. Moreover, by Lemma 6(ii) there are separated multigraphs with n vertices and $\Theta(n^2)$ edges, any two of which cross at most once. Hence, the term e^3/n^2 is best-possible by Lemma 1.

For $b = 3$ we have $x(b) = 1/(b-1) = 0.5$. Thus Theorem 2 for $D = D_{\mathrm{sep}}$ gives an absolute constant $\alpha > 0$ such that for every n-vertex e-edge separated multigraph we have $\mathrm{cr}(G) \geq \alpha e^{x(b)+2}/n^{x(b)+1} = \alpha e^{2.5}/n^{1.5}$, provided $e > (k_1 + 1)n = 4n$. Moreover, by Lemma 6(i) there are separated multigraphs with n vertices and $\Theta(n^3)$ edges, any two of which cross at most twice. Hence, the term $e^{2.5}/n^{1.5}$ is best-possible by Lemma 1.

4 Other Crossing Lemma Variants

We use the generalized Crossing Lemma (Theorem 2) to reprove existing variants of the Crossing Lemma due to Székely and Pach, Spencer, Tóth, respectively.

4.1 Low Multiplicity

Here we consider for fixed $m \geq 1$ the drawing style D_m which is characterized by the absence of $m+1$ pairwise parallel edges. In particular, any n-vertex multigraph G in drawing style D_m has at most $m\binom{n}{2}$ edges, i.e., **(P3)** holds for D_m with $b = 2$ and $k_3 = m$. Moreover, if G is crossing-free on n vertices and e edges, then $e \leq 3mn$, i.e., **(P1)** holds for D_m with $k_1 = 3m$.

Finally, we claim that **(P2)** holds for D_m with k_2 being independent of m. To this end, let G be any n-vertex e-edge multigraph in drawing style D_m. As already noted by Székely [6], we can reroute all but one edge in each bundle in such a way that in the resulting multigraph G' every lens is empty, no two adjacent edges cross, and $\mathrm{cr}(G') \leq \mathrm{cr}(G)$. (Simply route every edge very closely to its parallel copy with the least crossings.) Clearly, G' has drawing style D_m.

Now, we place a new vertex in each lens of G', giving a multigraph G'' with $n'' \leq n + e$ vertices and $e'' = e$ edges, which is in the separated drawing style D. By Lemma 4, there is an absolute constant k such that

$$b_D(G'') \leq k\sqrt{\mathrm{cr}(G'') + \Delta(G'') \cdot e'' + n''}.$$

As $b_{D_m}(G) \leq b_D(G'')$, $\mathrm{cr}(G'') = \mathrm{cr}(G') \leq \mathrm{cr}(G)$, $\Delta(G'') = \Delta(G)$, and $\Delta(G)+1 \leq 2\Delta(G)$ we conclude that

$$b_{D_m}(G) \leq 2k\sqrt{\mathrm{cr}(G) + \Delta(G) \cdot e + n}.$$

In other words, **(P2)** holds for drawing style D_m with an absolute constant $k_2 = 2k$ that is independent of m.

Note that for $b = 2$, we have $x(b) = 1$. We conclude with Theorem 2 and (1) that there is an absolute constant α' such that for every m and every n-vertex e-edge multigraph G in drawing style D_m we have

$$\mathrm{cr}(G) \geq \alpha' \cdot \frac{1}{k_3^x(b)} \cdot \frac{e^{x(b)+2}}{n^{x(b)+1}} = \alpha' \cdot \frac{e^3}{mn^2}, \qquad \text{provided } e > (3m+1)n,$$

which is the statement of Theorem B.

4.2 High Girth

Theorem D (Pach, Spencer, Tóth [4]) *For any $r \geq 1$ there is an absolute constant $\alpha_r > 0$ such that for any n-vertex e-edge graph G of girth larger than $2r$ we have*

$$\mathrm{cr}(G) \geq \alpha_r \cdot \frac{e^{r+2}}{n^{r+1}}, \qquad \textit{provided } e > 4n.$$

Here we consider for fixed $r \geq 1$ the drawing style D_r which is characterized by the absence of cycles of length at most $2r$. In particular, any multigraph G in drawing style D_r has neither loops nor multiple edges. Hence **(P1)** holds for drawing style D_r with $k_1 = 3$. Secondly, drawing style D_r is more restrictive than the branching drawing style and thus also **(P2)** holds for D_r. Moreover, any n-vertex graph in drawing style D_r has $O(n^{1+1/r})$ edges [2], i.e., **(P3)** holds for D_r with $b = 1+1/r$. Finally, D_r is obviously a monotone and split-compatible drawing style.

Thus with $x(b) = 1/(b-1) = r$, Theorem 2 immediately gives

$$\mathrm{cr}(G) \geq \alpha_r \cdot \frac{e^{r+2}}{n^{r+1}}, \qquad \text{provided } e > 4n$$

for any n-vertex e-edge multigraph in drawing style D_r, which is the statement of Theorem D.

5 Conclusions

Let G be a topological multigraph with n vertices and $e > 4n$ edges. We have shown that $\mathrm{cr}(G) \geq \alpha e^3/n^2$ if G is separated and locally starlike, which generalizes the result for branching multigraphs [5], which are additionally single-crossing. Moreover, if G is only separated, then the lower bound drops to $\mathrm{cr}(G) \geq$

$\alpha e^{2.5}/n^{1.5}$, which is tight up to the constant factor, too. It remains open to determine a best-possible Crossing Lemma for separated and single-crossing multigraphs. This would follow from our generalized Crossing Lemma (Theorem 2), where the missing ingredient is the determination of the smallest b such that every separated and single-crossing multigraph G on n vertices has $O(n^b)$ edges. It is easy to see that the maximum degree $\Delta(G)$ may be as high as $(n-1)(n-2)$, but we suspect that any such G has $O(n^2)$ edges.

Acknowledgements

This project initiated in the Dagstuhl seminar “Beyond-Planar Graphs: Algorithmics and Combinatorics”, November 2016. We would like to thank all participants, especially Stefan Felsner, Vincenzo Roselli, and Pavel Valtr, for fruitful discussions.

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