Double-normal pairs in space

János Pach[*]
EPFL Lausanne and
Rényi Institute, Budapest
pach@cims.nyu.edu

Konrad J. Swanepoel
Department of Mathematics,
London School of Economics and Political Science,
Houghton Street, London WC2A 2AE, United Kingdom
k.swanepoel@lse.ac.uk

Abstract

A *double-normal pair* of a finite set S of points that spans $\mathbb{R}^d$ is a pair of points $\{\boldsymbol{p},\boldsymbol{q}\}$ from S such that S lies in the closed strip bounded by the hyperplanes through $\boldsymbol{p}$ and $\boldsymbol{q}$ perpendicular to $\boldsymbol{pq}$. A double-normal pair $\{\boldsymbol{p},\boldsymbol{q}\}$ is *strict* if $S\setminus\{\boldsymbol{p},\boldsymbol{q}\}$ lies in the open strip. The problem of estimating the maximum number $N_d(n)$ of double-normal pairs in a set of n points in $\mathbb{R}^d$, was initiated by Martini and Soltan (2006).

It was shown in a companion paper that in the plane, this maximum is $3\lfloor n/2\rfloor$, for every $n>2$. For $d\geq 3$, it follows from the Erdős-Stone theorem in extremal graph theory that $N_d(n)=\frac{1}{2}(1-1/k)n^2+o(n^2)$ for a suitable positive integer $k=k(d)$. Here we prove that $k(3)=2$ and, in general, $\lceil d/2\rceil\leq k(d)\leq d-1$. Moreover, asymptotically we have $\lim_{n\to\infty}k(d)/d=1$. The same bounds hold for the maximum number of strict double-normal pairs.

1 Introduction

Let V be a set of n points that spans $\mathbb{R}^d$. A *double-normal pair* of V is a pair of distinct points $\{\boldsymbol{p},\boldsymbol{q}\}$ in V such that V lies in the closed strip bounded by the hyperplanes $H_{\boldsymbol{p}}$ and $H_{\boldsymbol{q}}$ through $\boldsymbol{p}$ and $\boldsymbol{q}$, respectively, that are perpendicular to the line $\boldsymbol{pq}$. A double-normal pair $\{\boldsymbol{p},\boldsymbol{q}\}$ is *strict* if $V\setminus\{\boldsymbol{p},\boldsymbol{q}\}$ is disjoint from the hyperplanes $H_{\boldsymbol{p}}$ and $H_{\boldsymbol{q}}$. Define the *double-normal graph* of V as the graph on the vertex set V in which two vertices p

[*]Research partially supported by Swiss National Science Foundation Grants 200021-137574 and 200020-144531, by Hungarian Science Foundation Grant OTKA NN 102029 under the EuroGIGA programs ComPoSe and GraDR, and by NSF grant CCF-08-30272.

and q are joined by an edge if and only if $\{\boldsymbol{p}, \boldsymbol{q}\}$ is a double-normal pair. The number of edges of this graph, that is, the number of double-normal pairs induced by V is denoted by $N(V)$.

We define the *strict double-normal graph* of V analogously and denote its number of edges by $N'(V)$.

Martini and Soltan [10, Problems 3 and 4] asked for the maximum numbers $N_d(n)$ and $N'_d(n)$ of double-normal pairs and strict double-normal pairs of a set of n points in $\mathbb{R}^d$:

$$N_d(n) := \max_{\substack{V\subset\mathbb{R}^d\\|V|=n}} N(V)$$

and

$$N'_d(n) := \max_{\substack{V\subset\mathbb{R}^d\\|V|=n}} N'(V).$$

Clearly, we have $N(V) \geq N'(V)$ and $N_d(n) \geq N'_d(n)$. It is not difficult to see that $N'_2(n) = n$. In another paper [12] we show that $N_2(n) = 3\lfloor n/2\rfloor$. Here we only consider the case $d \geq 3$.

Theorem 1. *The maximum number of double-normal and strict double-normal pairs in a set of n points in $\mathbb{R}^3$ satisfy $N_3(n) = n^2/4 + o(n^2)$ and $N'_3(n) = n^2/4 + o(n^2)$.*

In fact, since the collection of double-normal graphs in Euclidean space is closed under the taking of induced subgraphs, the Erdős–Stone Theorem [3] implies that for each $d \in \mathbb{N}$, there exist unique $k(d), k'(d) \in \mathbb{N}$ such that $N_d(n) = \frac{1}{2}(1 - \frac{1}{k(d)})n^2 + o(n^2)$ and $N'_d(n) = \frac{1}{2}(1 - \frac{1}{k'(d)})n^2 + o(n^2)$. The number $k(d)$ [resp. $k'(d)$] can also be characterised as the largest k such that complete k-partite graphs with arbitrarily many points in each class occur as subgraphs of double-normal [resp. strictly double-normal] graphs in $\mathbb{R}^d$. Theorem 1 states that $k(3) = k'(3) = 2$ and is a special case of the next theorem.

Theorem 2. *For each d, there exist unique integers $k(d), k'(d) \geq 1$ such that $N_d(n)$, the maximum number of double-normal pairs, and $N'_d(n)$, the maximum number of strict double-normal pairs in a set of n points in $\mathbb{R}^d$, satisfy*

$$N_d(n) = \frac{1}{2}\Big(1 - \frac{1}{k(d)}\Big)n^2 + o(n^2)$$

and

$$N'_d(n) = \frac{1}{2}\Big(1 - \frac{1}{k'(d)}\Big)n^2 + o(n^2).$$

For any $d \geq 3$, we have

$$\lceil d/2\rceil \leq k'(d) \leq k(d) \leq d - 1.$$

Asymptotically, as $d \to \infty$*, we have*

$$k(d) \geq k'(d) \geq d - O(\log d).$$

Although this theorem gives the exact values $k(3) = k'(3) = 2$, we do not know whether $k(4)$ or $k'(4)$ equals 2 or 3.

Two notions related to double-normal pairs have been studied before. We define a *diameter pair* of S to be a pair of points $\{\boldsymbol{p}, \boldsymbol{q}\}$ in S such that $|\boldsymbol{pq}| = \operatorname{diam}(S)$. Note that a diameter pair is also a strictly double-normal pair. The maximum number of diameter pairs in a set of n points is known for all $d \geq 2$, and in the case of $d \geq 4$, if n is sufficiently large [1, 4, 5, 13, 14, 6]. We call a pair $\boldsymbol{pq}$ of a set $S \subset \mathbb{R}^d$ *antipodal* if there exist parallel hyperplanes H_1 and H_2 through $\boldsymbol{p}$ and $\boldsymbol{q}$, respectively, such that S lies in the closed strip bounded by the hyperplanes. The pair is called *strictly antipodal* if there exist parallel hyperplanes through $\boldsymbol{p}$ and $\boldsymbol{q}$ such that $S \setminus \{\boldsymbol{p}, \boldsymbol{q}\}$ lies in the open strip bounded by the hyperplanes. Clearly, a (strictly) double-normal pair of a set is also a (strictly) antipodal pair. The problem of determining the asymptotic behaviour of the maximum number of antipodal or strictly antipodal pairs in a set of n points is open already in $\mathbb{R}^3$. For a thorough discussion of antipodal pairs, see the series of papers [7, 8, 9].

The paper is structured as follows. In Section 2, we collect some geometric lemmas on double-normal pairs. They are applied in Section 3 together with a Ramsey-type argument to derive the upper bound of Theorem 2 (Theorem 7). Finally, in Section 4 we show the two lower bounds of Theorem 2 (Corollaries 10 and 16). The asymptotic lower bound follows from a random construction closely related to the construction by Erdős and Füredi [2] of strictly antipodal sets of size exponential in the dimension.

We use the following notation. The inner product of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ is denoted by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$, the linear span of $S \subset \mathbb{R}^d$ by $\operatorname{lin} S$, the convex hull of S by $\operatorname{conv} S$, the diameter of S by $\operatorname{diam}(S)$, the cardinality of a finite set S by $|S|$, and the complete k-partite graph with N vertices in each class by $K_k(N)$. An angle with vertex $\boldsymbol{b}$ and sides $\boldsymbol{ba}$ and $\boldsymbol{bc}$ is denoted by $\angle \boldsymbol{abc}$, which we also use to denote its angular measure. All angles in this paper have angular measure in the range $(0, \pi)$. The Euclidean distance between $\boldsymbol{p}$ and $\boldsymbol{q}$ is denoted $|\boldsymbol{pq}|$.

2 Geometric properties of the double-normal relation

Here we collect some elementary geometric properties of double-normals pairs. They will be used in the next section where we find upper bounds to $k(d)$.

If a unit vector $\boldsymbol{u}$ is almost orthogonal to two given unit vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, then $\boldsymbol{u}$ is still almost orthogonal to any unit vector in the span of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, with an error that becomes worse the closer $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are to each other. The next lemma quantifies this observation.

Lemma 3. *Let $\boldsymbol{u}, \boldsymbol{u}_1, \boldsymbol{u}_2$ be unit vectors with $\boldsymbol{u}_1 \neq \pm\boldsymbol{u}_2$, such that for some $\varepsilon_1, \varepsilon_2 > 0$, $|\langle \boldsymbol{u}, \boldsymbol{u}_1\rangle| \leq \varepsilon_1$ and $|\langle \boldsymbol{u}, \boldsymbol{u}_2\rangle| \leq \varepsilon_2$. Then for any unit vector $\boldsymbol{v} \in \operatorname{lin}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ we have $|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| < (\varepsilon_1 + \varepsilon_2)/\sin\theta$, where $\theta \in (0, \pi)$ satisfies $\langle \boldsymbol{u}_1, \boldsymbol{u}_2\rangle = \cos\theta$.*

Proof. Let $\boldsymbol{u}'$ be the orthogonal projection of $\boldsymbol{u}$ onto the plane $\operatorname{lin}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$. Then the quantity $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{u}', \boldsymbol{v}\rangle$ is maximised when $\boldsymbol{v}$ is a positive multiple of $\boldsymbol{u}'$, and then $|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| = |\boldsymbol{o}\boldsymbol{u}'|$. It follows from the hypotheses that $\boldsymbol{u}'$ lies in the parallelogram P symmetric around $\boldsymbol{o}$ with sides perpendicular to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, respectively, and with the sides perpendicular to $\boldsymbol{u}_i$ at distance $2\varepsilon_i$, $i = 1, 2$. The sides of P form an angle of θ, and their lengths are $2\varepsilon_1/\sin\theta$ and $2\varepsilon_2/\sin\theta$. The maximum value of $|\boldsymbol{o}\boldsymbol{u}'|$ is attained at a vertex of the parallelogram P, that is, $|\boldsymbol{o}\boldsymbol{u}'|$ is at most half the largest diagonal of P. By the law of cosines, half a diagonal of P has length

$$\sqrt{\frac{\varepsilon_1^2}{\sin^2\theta} + \frac{\varepsilon_2^2}{\sin^2\theta} \pm 2\frac{\varepsilon_1\varepsilon_2}{\sin^2\theta}\cos\theta}$$
$$< \sqrt{\frac{\varepsilon_1^2}{\sin^2\theta} + \frac{\varepsilon_2^2}{\sin^2\theta} + 2\frac{\varepsilon_1\varepsilon_2}{\sin^2\theta}} = \frac{\varepsilon_1 + \varepsilon_2}{\sin\theta}. \qquad \square$$

Suppose that $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, $\boldsymbol{y}_3$ are collinear, with $\boldsymbol{y}_2$ between $\boldsymbol{y}_1$ and $\boldsymbol{y}_3$, and that $\boldsymbol{x}\boldsymbol{y}_2$ is a double-normal pair in some set that contains $\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3$. Then, since the segment $\boldsymbol{y}_1\boldsymbol{y}_3$ has to lie in the half-space through $\boldsymbol{y}_2$ with normal $\boldsymbol{y}_2\boldsymbol{x}$, it follows that $\boldsymbol{y}_1\boldsymbol{y}_3$ lies in the boundary of this half-space. That is, $\boldsymbol{x}\boldsymbol{y}_2 \perp \boldsymbol{y}_1\boldsymbol{y}_2$. If $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, $\boldsymbol{y}_3$ are close to collinear, then intuitively $\boldsymbol{y}_1\boldsymbol{y}_2$ will still be close to orthogonal to $\boldsymbol{x}\boldsymbol{y}_2$. This is the content of the next lemma.

Lemma 4. *Let $\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3$ be different points from $V \subset \mathbb{R}^d$, with $\boldsymbol{x}\boldsymbol{y}_2$ a double-normal pair in V. Let $\varepsilon > 0$ and suppose that $\angle \boldsymbol{y}_1\boldsymbol{y}_2\boldsymbol{y}_3 > \pi - \varepsilon$. Let $\boldsymbol{u}$ be a unit vector parallel to $\boldsymbol{y}_1\boldsymbol{y}_2$ and $\boldsymbol{v}$ a unit vector parallel to $\boldsymbol{x}\boldsymbol{y}_2$. Then $|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| < \varepsilon$.*

Proof. Without loss of generality, $\varepsilon < \pi/2$. Note that $\angle \boldsymbol{x}\boldsymbol{y}_2\boldsymbol{y}_1, \angle \boldsymbol{x}\boldsymbol{y}_2\boldsymbol{y}_3 \leq \pi/2$. Since also

$$\pi - \varepsilon < \angle \boldsymbol{y}_1\boldsymbol{y}_2\boldsymbol{y}_3 \leq \angle \boldsymbol{y}_1\boldsymbol{y}_2\boldsymbol{x} + \angle \boldsymbol{x}\boldsymbol{y}_2\boldsymbol{y}_3 \leq \angle \boldsymbol{y}_1\boldsymbol{y}_2\boldsymbol{x} + \pi/2,$$

we obtain

$$\pi/2 - \varepsilon < \angle \boldsymbol{y}_1\boldsymbol{y}_2\boldsymbol{x} \leq \pi/2,$$

and it follows that

$$|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| = \cos\angle \boldsymbol{y}_1\boldsymbol{y}_2\boldsymbol{x} < \cos(\pi/2 - \varepsilon) = \sin\varepsilon < \varepsilon. \qquad \square$$

Consider the situation where $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, $\boldsymbol{y}_3$ are "almost" collinear with $\boldsymbol{y}_2$ the "middle" point, but now there are two double-normal pairs $\boldsymbol{x}_1\boldsymbol{y}_2$ and

$\boldsymbol{x}_2\boldsymbol{y}_2$ in a set that contains $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3$. Then $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3$ all lie inside the wedge W formed by the intersection of the half-spaces H_1 and H_2 through $\boldsymbol{y}_2$ with normals $\boldsymbol{x}_1 - \boldsymbol{y}_2$ and $\boldsymbol{x}_2 - \boldsymbol{y}_2$, respectively. If $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, $\boldsymbol{y}_3$ are collinear with $\boldsymbol{y}_2$ between $\boldsymbol{y}_1$ and $\boldsymbol{y}_3$, then necessarily $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3$ all lie on the "ridge" $\operatorname{bd} H_1 \cap \operatorname{bd} H_2$ of the wedge W, and $\boldsymbol{y}_1\boldsymbol{y}_2$ is orthogonal to the plane Π through $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{y}_2$. If $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, $\boldsymbol{y}_3$ are close to collinear, then intuitively $\boldsymbol{y}_1\boldsymbol{y}_2$ will still be close to orthogonal to Π. The next lemma quantifies this intuition. It is an immediate consequence of Lemmas 3 and 4.

Lemma 5. *Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3$ be different points in $V \subset \mathbb{R}^d$, with $\boldsymbol{x}_1\boldsymbol{y}_2$ and $\boldsymbol{x}_2\boldsymbol{y}_2$ double-normal pairs in V. Let $\varepsilon > 0$. Suppose that $\angle \boldsymbol{y}_1\boldsymbol{y}_2\boldsymbol{y}_3 > \pi - \varepsilon$. Then for any unit vector $\boldsymbol{u}$ parallel to the line $\boldsymbol{y}_1\boldsymbol{y}_2$ and any unit vector $\boldsymbol{v}$ parallel to the plane $\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{y}_2$ we have $|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| < 2\varepsilon/\sin\angle \boldsymbol{x}_1\boldsymbol{y}_2\boldsymbol{x}_2$.*

If the angle $\angle \boldsymbol{x}_1\boldsymbol{y}_2\boldsymbol{x}_2$ in the previous lemma is small, then the bound obtained may be too large to be useful. In the next lemma, we show that we can still obtain a small upper bound if $|\boldsymbol{y}_1\boldsymbol{y}_2|$ is much smaller than $|\boldsymbol{x}_1\boldsymbol{x}_2|$. We need four double-normal pairs instead of the two required by Lemma 5, but we do not need $\boldsymbol{y}_3$.

Lemma 6. *Let $\boldsymbol{x}_i\boldsymbol{y}_j$, $i, j = 1, 2$, be four double-normal pairs in a set $V \subset \mathbb{R}^d$ that contains $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2$. Let $\boldsymbol{u}$ be a unit vector parallel to $\boldsymbol{y}_1\boldsymbol{y}_2$ and $\boldsymbol{v}$ a unit vector parallel to the plane $\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{y}_2$. Then*

$$|\langle \boldsymbol{u}, \boldsymbol{v}\rangle| \leq \frac{\sqrt{2}}{\cos^2\angle \boldsymbol{x}_1\boldsymbol{y}_2\boldsymbol{x}_2} \frac{|\boldsymbol{y}_1\boldsymbol{y}_2|}{|\boldsymbol{x}_1\boldsymbol{x}_2|}.$$

Proof. Let $\boldsymbol{u} := |\boldsymbol{y}_1\boldsymbol{y}_2|^{-1}(\boldsymbol{y}_1 - \boldsymbol{y}_2)$, $\boldsymbol{u}_1 := |\boldsymbol{x}_1\boldsymbol{y}_2|^{-1}(\boldsymbol{x}_1 - \boldsymbol{y}_2)$ and $\boldsymbol{u}_2 := |\boldsymbol{x}_1\boldsymbol{x}_2|^{-1}(\boldsymbol{x}_1 - \boldsymbol{x}_2)$. Then $\langle \boldsymbol{u}_1, \boldsymbol{u}_2\rangle = \cos\theta$ where $\theta := \angle \boldsymbol{x}_2\boldsymbol{x}_1\boldsymbol{y}_2$.

Since the angles $\angle \boldsymbol{x}_1\boldsymbol{y}_1\boldsymbol{y}_2$, $\angle \boldsymbol{x}_1\boldsymbol{y}_2\boldsymbol{y}_1$, $\angle \boldsymbol{x}_2\boldsymbol{y}_2\boldsymbol{y}_1$ are non-obtuse, we obtain

$$\langle \boldsymbol{x}_1 - \boldsymbol{y}_1, \boldsymbol{y}_2 - \boldsymbol{y}_1\rangle \geq 0, \tag{1}$$

$$\langle \boldsymbol{x}_1 - \boldsymbol{y}_2, \boldsymbol{y}_1 - \boldsymbol{y}_2\rangle \geq 0, \tag{2}$$

and

$$\langle \boldsymbol{y}_2 - \boldsymbol{x}_2, \boldsymbol{y}_2 - \boldsymbol{y}_1\rangle \geq 0. \tag{3}$$

From (1) we obtain $\langle \boldsymbol{x}_1 - \boldsymbol{y}_2, \boldsymbol{y}_2 - \boldsymbol{y}_1\rangle \geq -|\boldsymbol{y}_1\boldsymbol{y}_2|^2$, that is,

$$\langle \boldsymbol{u}, \boldsymbol{u}_1\rangle \leq |\boldsymbol{y}_2\boldsymbol{y}_1|/|\boldsymbol{x}_1\boldsymbol{y}_2| =: \varepsilon_1.$$

From (2), $\langle \boldsymbol{u}, \boldsymbol{u}_1\rangle \geq 0$. Next, add (1) and (3) to obtain $\langle \boldsymbol{x}_2 - \boldsymbol{x}_1, \boldsymbol{y}_2 - \boldsymbol{y}_1\rangle \leq |\boldsymbol{y}_1\boldsymbol{y}_2|^2$, that is,

$$\langle \boldsymbol{u}, \boldsymbol{u}_2\rangle \leq |\boldsymbol{y}_1\boldsymbol{y}_2|/|\boldsymbol{x}_1\boldsymbol{x}_2| =: \varepsilon_2.$$

The analogues of (1) and (3) with $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ interchanged similarly give $-\langle \boldsymbol{u}, \boldsymbol{u}_2 \rangle \leq \varepsilon_2$. By Lemma 3, for any unit vector $\boldsymbol{v}$ parallel to the plane Π through $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{y}_2$, that is, with $\boldsymbol{v} \in \operatorname{lin}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, we have

$$|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| \leq \frac{\varepsilon_1 + \varepsilon_2}{\sin\theta}. \tag{4}$$

By the law of sines in $\triangle \boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{y}_2$,

$$\frac{\varepsilon_1}{\varepsilon_2} = \frac{|\boldsymbol{x}_1\boldsymbol{x}_2|}{|\boldsymbol{x}_1\boldsymbol{y}_2|} = \frac{\sin\alpha}{\sin\varphi},$$

where $\varphi := \angle \boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{y}_2$ and $\alpha := \angle \boldsymbol{x}_1\boldsymbol{y}_2\boldsymbol{x}_2$. It follows from (4) that

$$|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| \leq \frac{\varepsilon_2}{\sin\theta}\left(1 + \frac{\sin\alpha}{\sin\varphi}\right).$$

Since $\alpha, \theta, \varphi \leq \pi/2$ and $\alpha + \theta + \varphi = \pi$, we have

$$\sin\theta, \sin\varphi \geq \sin(\pi/2 - \alpha) = \cos\alpha,$$

hence

$$\begin{aligned}
|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| &\leq \frac{\varepsilon_2}{\cos\alpha}\left(1 + \frac{\sin\alpha}{\cos\alpha}\right) = \frac{\varepsilon_2}{\cos^2\alpha}(\cos\alpha + \sin\alpha) \\
&\leq \frac{\varepsilon_2}{\cos^2\alpha}\sqrt{2} = \frac{\sqrt{2}}{\cos^2\alpha}\frac{|\boldsymbol{y}_1\boldsymbol{y}_2|}{|\boldsymbol{x}_1\boldsymbol{x}_2|}.
\end{aligned} \qquad \square$$

3 Upper bound on the number of double-normal pairs

Recall that $k(d)$ denotes the largest k such that for each $N \in \mathbb{N}$, the complete k-partite graph with N vertices in each class, $K_k(N)$, is a subgraph of some double-normal graph in $\mathbb{R}^d$.

Theorem 7. *For all $d \geq 3$, we have $k(d) \leq d - 1$.*

This theorem is a straightforward consequence of the following technical result.

Proposition 8. *There exist a family of $k = k(d)$ not necessarily distinct points $\{\boldsymbol{p}_1, \dots, \boldsymbol{p}_k\}$ and a family of k^2 not necessarily distinct unit vectors $\{\boldsymbol{u}_{i,j} : 1 \leq i, j \leq k\}$, all in $\mathbb{R}^d$, such that the following holds:*

(5) *$\{\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_k\}$ has at least two distinct points and no obtuse angles.*

(6) *$\{\boldsymbol{u}_{1,1}, \boldsymbol{u}_{2,2}, \dots, \boldsymbol{u}_{k,k}\}$ is an orthogonal set.*

(7) *If $i \neq j$, then $\boldsymbol{u}_{i,j} = -\boldsymbol{u}_{j,i}$.*

(8) *If $\boldsymbol{p}_i \neq \boldsymbol{p}_j$, then $\boldsymbol{u}_{i,j} = |\boldsymbol{p}_j\boldsymbol{p}_i|^{-1}(\boldsymbol{p}_j - \boldsymbol{p}_i)$.*

(9) *For any distinct i, j, $\boldsymbol{u}_{i,i}$ is orthogonal to $\boldsymbol{u}_{i,j}$.*

(10) *Each $\boldsymbol{u}_{i,i}$ is orthogonal to the subspace* $\operatorname{lin}\{\boldsymbol{p}_j - \boldsymbol{p}_1 : j = 2, \dots, k\}$.

(11) *If $\boldsymbol{p}_i = \boldsymbol{p}_{i'} \neq \boldsymbol{p}_j$, then $\boldsymbol{u}_{i,i'}$ is orthogonal to $\boldsymbol{u}_{i,j} = \boldsymbol{u}_{i',j}$.*

Algorithm 1: Pruning the sets V_i

```
for i = 1 to k do
    (Note that here |V_j| = 2t^{k-i} + 1 for all j ≥ i)
    relabel V_i, ..., V_k such that diam(V_i) = max {diam(V_j) : j > i}
    for j = i + 1 to k do
        find V'_j ⊆ V_j such that |V'_j| = 2t^{k-i-1} + 1
            and diam(V'_j) ≤ ε diam(V_j);
        replace V_j by V'_j;
```

Proof. The proof consists of three steps.

Step 1. We will use a geometric Ramsey-type result from [11] and the pigeon-hole principle to show that for any $\varepsilon > 0$ there exists N such that for any $K_k(N)$ with classes $V_1, \dots, V_k$ contained in some double-normal graph in $\mathbb{R}^d$, there exist points $\boldsymbol{a}_i, \boldsymbol{b}_i, \boldsymbol{c}_i \in V_i$ $(i = 1, \dots, k)$ such that

$$\angle \boldsymbol{a}_i\boldsymbol{b}_i\boldsymbol{c}_i > \pi - \varepsilon, \quad i = 1, \dots, k, \tag{12}$$

$$|\boldsymbol{a}_{i+1}\boldsymbol{c}_{i+1}| \leq \varepsilon |\boldsymbol{a}_i\boldsymbol{c}_i|, \quad i = 1, \dots, k-1, \tag{13}$$

$$|\boldsymbol{a}_i\boldsymbol{b}_i| \geq \frac{1}{2}|\boldsymbol{a}_i\boldsymbol{c}_i|, \quad i = 1, \dots, k. \tag{14}$$

Step 2. We use the results from Section 2 to show that if we set $\boldsymbol{u}_{i,i} := |\boldsymbol{a}_i\boldsymbol{b}_i|^{-1}(\boldsymbol{a}_i - \boldsymbol{b}_i)$ and $\boldsymbol{u}_{i,j} := |\boldsymbol{b}_j\boldsymbol{b}_i|^{-1}(\boldsymbol{b}_j - \boldsymbol{b}_i)$, then

$$|\langle \boldsymbol{u}_{i,i}, \boldsymbol{u}_{i,j}\rangle| < \varepsilon, \quad i, j = 1, \dots, k, \ i \neq j. \tag{15}$$

$$|\langle \boldsymbol{u}_{i,i}, \boldsymbol{u}_{j,j}\rangle| < 4\varepsilon, \quad i, j = 1, \dots, k, \ i \neq j \tag{16}$$

Step 3. The proposition will follow by setting $\varepsilon := 1/n$ and taking subsequences of the sequences $\boldsymbol{a}_i^{(n)}, \boldsymbol{b}_i^{(n)}, \boldsymbol{c}_i^{(n)}$, $i = 1, \dots, k$, such that $\boldsymbol{b}_i^{(n)}$ converges to $\boldsymbol{p}_i$, and each $\boldsymbol{u}_{i,j}^{(n)}$ converges, as $n \to \infty$. The details follow.

Let $\varepsilon > 0$ be given. Write $t := \lceil (\varepsilon \cos \varepsilon)^{-1} \rceil$. In **Step 1**, applying [11, Theorem 4] we first choose a sufficiently large N depending only on ε and d such that each class V_i of any $K_k(N)$ contained in a double-normal graph in $\mathbb{R}^d$ has a subset V_i' of size $2t^{k-1} + 1$ such that for any $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ from the same V_i' with $\boldsymbol{a} \neq \boldsymbol{b}$ and $\boldsymbol{c} \neq \boldsymbol{d}$, the angle between the lines $\boldsymbol{ab}$ and $\boldsymbol{cd}$ is less than ε. We now replace the original V_i by V_i'. If we assume $\varepsilon < \pi/3$, we obtain a natural linear ordering (more precisely, a betweenness relation) on the points of each V_i, by defining for each $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V_i$ that $\boldsymbol{y}$ is *between* $\boldsymbol{x}$ and $\boldsymbol{z}$ if $\angle \boldsymbol{xyz} > \pi - \varepsilon$. Then $|\boldsymbol{yx}| < |\boldsymbol{zx}|$ whenever $\boldsymbol{y}$ is between $\boldsymbol{x}$ and $\boldsymbol{z}$.

Next we run Algorithm 1 on $V_1, \dots, V_k$. Note that at the start of the outer **for** loop, $|V_j| = 2t^{k-i}+1$ for all $j = i, \dots, k$. That we can find a V_j' as required inside the inner **for** loop, is seen as follows. Write $V_j := \{\boldsymbol{p}_1, \dots, \boldsymbol{p}_{2t^{k-i}+1}\}$

with the points in their natural order (where $\boldsymbol{p}_j$ is between $\boldsymbol{p}_i$ and $\boldsymbol{p}_k$ if $\angle \boldsymbol{p}_i\boldsymbol{p}_j\boldsymbol{p}_k > \pi - \varepsilon$). Let $\boldsymbol{p}_i'$ be the orthogonal projection of $\boldsymbol{p}_i$ onto the line ℓ through $\boldsymbol{p}_1$ and $\boldsymbol{p}_{2t^{k-i}+1}$. Since $\varepsilon < \pi/2$, the points $\boldsymbol{p}_i'$ are in order on ℓ, and

$$\begin{aligned}|\boldsymbol{p}_1\boldsymbol{p}_{2t^{k-i}+1}| &= |\boldsymbol{p}_1'\boldsymbol{p}_{2t^{k-i}+1}'| \\ &= \sum_{s=1}^{t} |\boldsymbol{p}_{2t^{k-i-1}(s-1)+1}'\boldsymbol{p}_{2t^{k-i-1}s+1}'| \\ &> \cos\varepsilon \sum_{s=1}^{t} |\boldsymbol{p}_{2t^{k-i-1}(s-1)+1}\boldsymbol{p}_{2t^{k-i-1}s+1}|,\end{aligned}$$

where the last inequality holds, because the angle between ℓ and the line through any two $\boldsymbol{p}_i$ is less than ε. Thus,

$$\begin{aligned}&\frac{1}{t}\sum_{s=1}^{t} |\boldsymbol{p}_{2t^{k-i-1}(s-1)+1}\boldsymbol{p}_{2t^{k-i-1}s+1}| \\ &< \frac{1}{t\cos\varepsilon}|\boldsymbol{p}_1\boldsymbol{p}_{2t^{k-i}+1}| < \varepsilon|\boldsymbol{p}_1\boldsymbol{p}_{2t^{k-i}+1}|.\end{aligned}$$

It follows that for some $s \in \{1, \ldots, t\}$,

$$|\boldsymbol{p}_{2t^{k-i-1}(s-1)+1}\boldsymbol{p}_{2t^{k-i-1}s+1}| < \varepsilon|\boldsymbol{p}_1\boldsymbol{p}_{2t^{k-i}+1}|.$$

Let $V_j' := \left\{\boldsymbol{p}_{2t^{k-i-1}(s-1)+1}, \ldots, \boldsymbol{p}_{2t^{k-i-1}s+1}\right\}$. Then $\left|V_j'\right| = 2t^{k-i-1} + 1$ and

$$\operatorname{diam}(V_j') < \varepsilon|\boldsymbol{p}_1\boldsymbol{p}_{2t^{k-i}+1}| = \varepsilon \operatorname{diam}(V_j).$$

When the algorithm is done, we have sets $V_1, \ldots, V_k$ such that $\operatorname{diam}(V_{i+1}) \geq \varepsilon \operatorname{diam}(V_i)$ for each $i = 1, \ldots, k-1$, and $|V_i| = 2t^{k-i} + 1 \geq 3$ for each $i = 1, \ldots, k$. Let $\boldsymbol{a}_i\boldsymbol{c}_i$ be a diameter of V_i and choose any $\boldsymbol{b}_i \in V_i \setminus \{\boldsymbol{a}_i, \boldsymbol{c}_i\}$. Then (12) and (13) hold. To ensure (14), exchange $\boldsymbol{a}_i$ and $\boldsymbol{c}_i$ if necessary such that $|\boldsymbol{a}_i\boldsymbol{b}_i| \geq |\boldsymbol{c}_i\boldsymbol{b}_i|$. Then (14) follows from the triangle inequality.

In **Step 2** we show (15) and (16). Let $1 \leq i, j \leq k$, $i \neq j$. Without loss of generality, $i < j$. Then (15) follows upon applying Lemma 4 with $\boldsymbol{x} := \boldsymbol{b}_i$, $\boldsymbol{y}_1 := \boldsymbol{a}_j$, $\boldsymbol{y}_2 := \boldsymbol{b}_j$, $\boldsymbol{y}_3 := \boldsymbol{c}_j$.

If $\angle \boldsymbol{a}_i\boldsymbol{b}_j\boldsymbol{b}_i \geq \pi/6$, then by Lemma 5 with $\boldsymbol{x}_1 := \boldsymbol{a}_i$, $\boldsymbol{x}_2 := \boldsymbol{b}_i$, $\boldsymbol{y}_1 := \boldsymbol{a}_j$, $\boldsymbol{y}_2 := \boldsymbol{b}_j$, $\boldsymbol{y}_3 := \boldsymbol{c}_j$,

$$|\langle \boldsymbol{u}_{i,i}, \boldsymbol{u}_{j,j}\rangle| < \frac{2\varepsilon}{\sin\angle \boldsymbol{a}_i\boldsymbol{b}_j\boldsymbol{b}_i} \leq \frac{2\varepsilon}{\sin\pi/6} = 4\varepsilon.$$

If $\angle \boldsymbol{a}_i\boldsymbol{b}_j\boldsymbol{b}_i < \pi/6$, then by Lemma 6 with $\boldsymbol{x}_1 := \boldsymbol{a}_i$, $\boldsymbol{x}_2 := \boldsymbol{b}_i$, $\boldsymbol{y}_1 := \boldsymbol{a}_j$, $\boldsymbol{y}_2 := \boldsymbol{b}_j$,

$$\begin{aligned}|\langle \boldsymbol{u}_{i,i}, \boldsymbol{u}_{j,j}\rangle| &< \frac{\sqrt{2}}{\cos^2\angle \boldsymbol{a}_i\boldsymbol{b}_j\boldsymbol{b}_i}\frac{|\boldsymbol{a}_j\boldsymbol{b}_j|}{|\boldsymbol{a}_i\boldsymbol{b}_i|} \\ &< \frac{\sqrt{2}}{\cos^2(\pi/6)}\frac{|\boldsymbol{a}_j\boldsymbol{c}_j|}{\frac{1}{2}|\boldsymbol{a}_i\boldsymbol{c}_i|} < (8\sqrt{2}/3)\varepsilon < 4\varepsilon,\end{aligned}$$

which shows (16).

In **Step 3**, we let $n \in \mathbb{N}$ be arbitrary, set $\varepsilon := 1/n$, and choose $\boldsymbol{a}_i^{(n)}$, $\boldsymbol{b}_i^{(n)}$, $\boldsymbol{c}_i^{(n)}$, $i = 1, \dots, k$, as in the first stage of the proof. We may assume, after translating and scaling each $\bigcup_{i=1}^k V_i^{(n)}$ if necessary, that $\left\{\boldsymbol{b}_1^{(n)}, \dots, \boldsymbol{b}_k^{(n)}\right\}$ has diameter 1 and is contained in the unit ball. Thus, we may pass to subsequences to assume that for each i, $\boldsymbol{b}_i^{(n)}$ converges to $\boldsymbol{p}_i$, say,

$$\boldsymbol{u}_{i,i}^{(n)} := |\boldsymbol{a}_i^{(n)}\boldsymbol{b}_i^{(n)}|^{-1}(\boldsymbol{a}_i^{(n)} - \boldsymbol{b}_i^{(n)})$$

converges to $\boldsymbol{u}_{i,i}$, say, and

$$\boldsymbol{u}_{i,j}^{(n)} := |\boldsymbol{b}_j^{(n)}\boldsymbol{b}_i^{(n)}|^{-1}(\boldsymbol{b}_j^{(n)} - \boldsymbol{b}_i^{(n)})$$

converges to $\boldsymbol{u}_{i,j}$, say. Then $\operatorname{diam}\{\boldsymbol{p}_1, \dots, \boldsymbol{p}_k\} = 1$, and since there are no obtuse angles in $\left\{\boldsymbol{b}_1^{(n)}, \dots, \boldsymbol{b}_k^{(n)}\right\}$, there will still be no obtuse angles between distinct elements of $\{\boldsymbol{p}_1, \dots, \boldsymbol{p}_k\}$. Thus, (5) holds. Also, (6) follows from (16), (7) from the definition of $\boldsymbol{u}_{i,j}^{(n)}$, (8) from the definitions of $\boldsymbol{u}_{i,j}^{(n)}$ and $\boldsymbol{p}_i$, and (9) from (15). Properties (8) and (9) immediately imply that $\boldsymbol{u}_{i,i}$ is orthogonal to $\boldsymbol{p}_i - \boldsymbol{p}_j$ for all $j \neq i$. Since the subspace $\operatorname{lin}\{\boldsymbol{p}_i - \boldsymbol{p}_j : j \neq i\}$ is the same for all i, we obtain (10).

Finally, suppose $\boldsymbol{p}_i = \boldsymbol{p}_{i'} \neq \boldsymbol{p}_j$. Since $\angle \boldsymbol{b}_i^{(n)}\boldsymbol{b}_j^{(n)}\boldsymbol{b}_{i'}^{(n)} \to 0$ as $n \to \infty$ and $\triangle \boldsymbol{b}_i\boldsymbol{b}_{i'}\boldsymbol{b}_j$ is not obtuse, we obtain that $\angle \boldsymbol{b}_i^{(n)}\boldsymbol{b}_{i'}^{(n)}\boldsymbol{b}_j^{(n)} \to \pi/2$ and $\angle \boldsymbol{b}_{i'}^{(n)}\boldsymbol{b}_i^{(n)}\boldsymbol{b}_j^{(n)} \to \pi/2$ as $n \to \infty$, giving $\boldsymbol{u}_{i,i'} \perp \boldsymbol{u}_{i,j}$. This shows (11). □

Proof of Theorem 7. Let $k = k(d)$. Consider the points $\boldsymbol{p}_1, \dots, \boldsymbol{p}_k$ and vectors $\boldsymbol{u}_{i,j}$, $1 \leq i,j \leq k$ given by Proposition 8. There exist distinct i and j such that $\boldsymbol{p}_i \neq \boldsymbol{p}_j$. By (6), the k unit vectors $\boldsymbol{u}_{1,1}, \dots, \boldsymbol{u}_{k,k}$ are pairwise orthogonal. By (10), they are also orthogonal to $\boldsymbol{p}_i - \boldsymbol{p}_j$, which is a multiple of $\boldsymbol{u}_{i,j}$ by (8). Thus, we have found $k+1$ pairwise orthogonal vectors. That is, $k(d) + 1 \leq d$. □

4 Constructions with many strict double-normal pairs

Theorem 9. *Let $m \geq 2$. Suppose that there exist m points $\boldsymbol{p}_1, \dots, \boldsymbol{p}_m \in \mathbb{R}^d$ and m unit vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m \in \mathbb{R}^d$ such that, for all triples of distinct i, j, k, the angle $\angle \boldsymbol{p}_i\boldsymbol{p}_j\boldsymbol{p}_k$ is acute, and*

$$\langle \boldsymbol{u}_i, \boldsymbol{p}_i - \boldsymbol{p}_j\rangle < \langle \boldsymbol{u}_i, \boldsymbol{p}_k - \boldsymbol{p}_j\rangle < \langle \boldsymbol{u}_i, \boldsymbol{p}_j - \boldsymbol{p}_i\rangle. \tag{17}$$

Then, for any $N \in \mathbb{N}$, there exists a strict double-normal graph in $\mathbb{R}^{d+m}$ containing a complete m-partite $K_m(N)$. In particular, $k'(d+m) \geq m$.

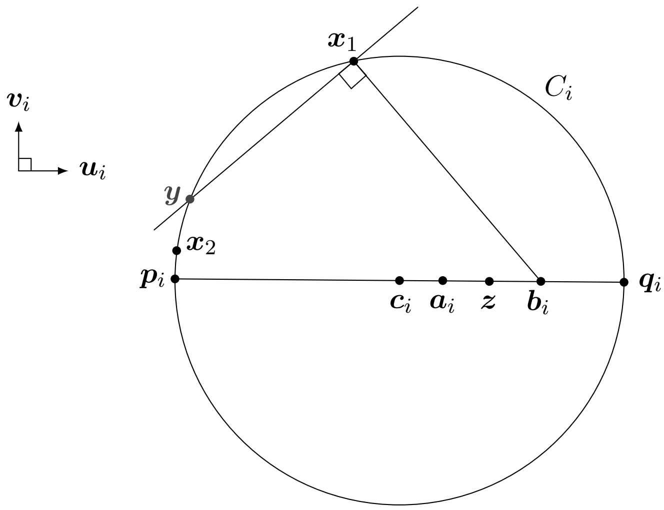

Figure 1: Constructing $V_i = \{\boldsymbol{x}_t : t \in \mathbb{N}\}$

Geometrically, (17) means that if we project the points $\boldsymbol{p}_1, \dots, \boldsymbol{p}_m$ orthogonally onto the line through $\boldsymbol{p}_i$ parallel to $\boldsymbol{u}_i$, then the projected points are on the ray from $\boldsymbol{p}_i$ in the direction of $\boldsymbol{u}_i$, and the furthest one is at less than twice the distance from $\boldsymbol{p}_i$ than the closest one (other than $\boldsymbol{p}_i$).

Proof. Identify $\mathbb{R}^d$ with the first d coordinates of $\mathbb{R}^{d+m}$, and let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbb{R}^{d+m}$ be pairwise orthogonal unit vectors that are also orthogonal to $\mathbb{R}^d$. We will construct countably infinite sets $V_1, \dots, V_m \subset \mathbb{R}^{d+m}$, with each V_i on a circular arc through $\boldsymbol{p}_i$ in the plane $\Pi_i := \boldsymbol{p}_i + \operatorname{lin}\{\boldsymbol{u}_i, \boldsymbol{v}_i\}$. Then we will verify that for any distinct i, j and any $\boldsymbol{x} \in V_i$ and $\boldsymbol{y} \in V_j$, $\boldsymbol{xy}$ is a strict double-normal pair of $\bigcup_i V_i$.

We will use a small $\varepsilon > 0$ that will depend only on the given points $\boldsymbol{p}_1, \dots, \boldsymbol{p}_m$ and vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m$. As the proof progresses, we will put finitely many constraints on ε, all depending only on the points $\boldsymbol{p}_i$ and vectors $\boldsymbol{u}_i$.

Let $\alpha_i := \min_{j \neq i} \langle \boldsymbol{u}_i, \boldsymbol{p}_j \rangle$ and $\beta_i := \max_j \langle \boldsymbol{u}_i, \boldsymbol{p}_j \rangle$. By condition (17), $\langle \boldsymbol{u}_i, \boldsymbol{p}_i \rangle - \alpha_i < \beta_i - \alpha_i < \alpha_i - \langle \boldsymbol{u}_i, \boldsymbol{p}_i \rangle$, hence $\langle \boldsymbol{u}_i, \boldsymbol{p}_i \rangle < \frac{1}{2}(\beta_i + \langle \boldsymbol{u}_i, \boldsymbol{p}_i \rangle) < \alpha_i$. We choose $\varepsilon > 0$ small enough so that $\frac{1}{2}(\beta_i + \varepsilon + \langle \boldsymbol{u}_i, \boldsymbol{p}_i \rangle) < \alpha_i - \varepsilon$ for all i. Choose any $r_i \in (\frac{1}{2}(\beta_i + \varepsilon + \langle \boldsymbol{u}_i, \boldsymbol{p}_i \rangle), \alpha_i - \varepsilon)$, and set $\boldsymbol{c}_i := \boldsymbol{p}_i + r_i \boldsymbol{u}_i$, $\boldsymbol{a}_i := \boldsymbol{p}_i + (\alpha_i - \varepsilon)\boldsymbol{u}_i$, $\boldsymbol{b}_i := \boldsymbol{p}_i + (\beta_i + \varepsilon)\boldsymbol{u}_i$, $\boldsymbol{q}_i := \boldsymbol{p}_i + 2r_i \boldsymbol{u}_i$ (Fig. 1). Denote the circle with centre $\boldsymbol{c}_i$ and radius r_i in the plane Π_i by C_i. Then $\boldsymbol{p}_i\boldsymbol{q}_i$ is a diameter of C_i parallel to $\boldsymbol{u}_i$, and $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ are strictly between $\boldsymbol{c}_i$ and $\boldsymbol{q}_i$. Choose any $\boldsymbol{x}_1 \in C_i \setminus \{\boldsymbol{p}_i\}$ such that $\angle \boldsymbol{x}_1 \boldsymbol{c}_i \boldsymbol{p}_i$ is acute. We will now recursively choose $\boldsymbol{x}_2, \boldsymbol{x}_3, \dots$ on the minor arc γ_i of C_i between $\boldsymbol{x}_1$ and $\boldsymbol{p}_i$ such that for any $\boldsymbol{z}$ on the segment $\boldsymbol{a}_i\boldsymbol{b}_i$, the angle $\angle \boldsymbol{z}\boldsymbol{x}_t\boldsymbol{x}_s$ is acute for all distinct $s, t \in \mathbb{N}$. Assume that for some $t \in \mathbb{N}$ we have already chosen $\boldsymbol{x}_1, \dots, \boldsymbol{x}_t \in \gamma_i$ with $\boldsymbol{x}_{s+1}$ between $\boldsymbol{x}_s$ and $\boldsymbol{p}_i$ for each $s = 1, \dots, t-1$, and such that $\angle \boldsymbol{z}\boldsymbol{x}_j\boldsymbol{x}_k$ is acute for all $1 \le j,k \le t$, $j \ne k$, and for all $\boldsymbol{z}$ on the segment $\boldsymbol{a}_i\boldsymbol{b}_i$. Since $\boldsymbol{p}_i\boldsymbol{x}_t\boldsymbol{q}_i$ is a right angle, $\angle \boldsymbol{p}_i\boldsymbol{x}_t\boldsymbol{b}_i$ is acute, and the line in Π_i through $\boldsymbol{x}_t$ and perpendicular to $\boldsymbol{b}_i\boldsymbol{x}_t$ intersects C_i in a point $\boldsymbol{y} \in \gamma_i$ between $\boldsymbol{x}_t$ and $\boldsymbol{p}_i$. Let $\boldsymbol{x}_{t+1}$ be any point on γ_i between $\boldsymbol{y}$ and $\boldsymbol{p}_i$. Now consider any $\boldsymbol{z}$ on the segment $\boldsymbol{a}_i\boldsymbol{b}_i$. We have to show that $\angle \boldsymbol{z}\boldsymbol{x}_{t+1}\boldsymbol{x}_s$ and $\boldsymbol{z}\boldsymbol{x}_s\boldsymbol{x}_{t+1}$ are acute for all $s = 1,\dots,t$. This can be simply seen as follows:

$$\angle \boldsymbol{z}\boldsymbol{x}_{t+1}\boldsymbol{x}_s \le \angle \boldsymbol{z}\boldsymbol{x}_{t+1}\boldsymbol{x}_t \le \angle \boldsymbol{c}_i\boldsymbol{x}_{t+1}\boldsymbol{x}_t < \pi/2$$

and

$$\angle \boldsymbol{z}\boldsymbol{x}_s\boldsymbol{x}_{t+1} \le \angle \boldsymbol{z}\boldsymbol{x}_t\boldsymbol{x}_{t+1} \le \angle \boldsymbol{b}_i\boldsymbol{x}_t\boldsymbol{x}_{t+1} < \angle \boldsymbol{b}_i\boldsymbol{x}_t\boldsymbol{y} = \pi/2.$$

Finally, let $V_i := \{\boldsymbol{x}_t : t \in \mathbb{N}\}$. Then $\operatorname{diam} V_i = |\boldsymbol{p}_i\boldsymbol{x}_1|$, which can be made arbitrarily small by choosing $\boldsymbol{x}_1$ close enough to $\boldsymbol{p}_i$. We can assume that all $\operatorname{diam}(V_i) < \varepsilon$. This finishes the construction.

Let $1 \le i < j \le m$, $\boldsymbol{x} \in V_i$ and $\boldsymbol{y} \in V_j$. We have to show that all $\boldsymbol{z} \in \bigcup_i V_i \setminus \{\boldsymbol{x},\boldsymbol{y}\}$ are in the open slab bounded by the hyperplanes through $\boldsymbol{x}$ and $\boldsymbol{y}$ orthogonal to $\boldsymbol{x}\boldsymbol{y}$. First consider the case where $\boldsymbol{z} \in V_k$, $k \ne i,j$. Since $\angle \boldsymbol{p}_i\boldsymbol{p}_j\boldsymbol{p}_k$ and $\angle \boldsymbol{p}_j\boldsymbol{p}_i\boldsymbol{p}_k$ are acute, $\langle \boldsymbol{p}_i - \boldsymbol{p}_j, \boldsymbol{p}_k - \boldsymbol{p}_j\rangle > 0$ and $\langle \boldsymbol{p}_j - \boldsymbol{p}_i, \boldsymbol{p}_k - \boldsymbol{p}_i\rangle > 0$. Noting that $|\boldsymbol{x}\boldsymbol{p}_i|, |\boldsymbol{y}\boldsymbol{p}_j|, |\boldsymbol{z}\boldsymbol{p}_k| < \varepsilon$, it follows that $\langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{z} - \boldsymbol{y}\rangle > 0$ and $\langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{z} - \boldsymbol{x}\rangle > 0$ if ε is sufficiently small, depending only on the given points. That is, $\boldsymbol{z}$ is in the open slab determined by $\boldsymbol{x}\boldsymbol{y}$.

Next consider the case where $\boldsymbol{z} \in V_i \cup V_j$. Without loss of generality, $\boldsymbol{z} \in V_i$. Then

$$\langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{z} - \boldsymbol{y}\rangle = \langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{z} - \boldsymbol{x}\rangle + |\boldsymbol{x}\boldsymbol{y}|^2 \ge -\varepsilon|\boldsymbol{x}\boldsymbol{y}| + |\boldsymbol{x}\boldsymbol{y}|^2 > 0,$$

as long as $\varepsilon < |\boldsymbol{x}\boldsymbol{y}|$. It remains to verify that $\langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{z} - \boldsymbol{x}\rangle > 0$. Denote the orthogonal projection of a point $\boldsymbol{p} \in \mathbb{R}^{d+m}$ onto the plane Π_i by $\boldsymbol{p}'$. Since $V_j \subset \Pi_j \subseteq \mathbb{R}^d + \operatorname{lin}\{\boldsymbol{v}_j\}$, it follows that $\boldsymbol{p}_j', \boldsymbol{y}' \in \boldsymbol{p}_i + \operatorname{lin}\{\boldsymbol{u}_i\}$. In particular, $\boldsymbol{p}_j'$ is also the orthogonal projection of $\boldsymbol{p}_j$ onto the line $\boldsymbol{p}_i + \operatorname{lin}\{\boldsymbol{u}_i\}$. By hypothesis, $\boldsymbol{p}_j' = \boldsymbol{p}_i + \lambda\boldsymbol{u}_i$ for some $\lambda \in [\alpha_i, \beta_i]$. Since $|\boldsymbol{p}_j'\boldsymbol{y}'| \le |\boldsymbol{p}_j\boldsymbol{y}| < \varepsilon$, it follows that $\boldsymbol{y}' = \boldsymbol{p}_i + \mu\boldsymbol{u}_i$ where $\mu \in [\alpha_i - \varepsilon, \beta_i + \varepsilon]$, that is, $\boldsymbol{y}'$ is on the segment $\boldsymbol{a}_i\boldsymbol{b}_i$. By construction, the angle $\angle \boldsymbol{y}'\boldsymbol{x}\boldsymbol{z}$ is acute, hence $\langle \boldsymbol{y} - \boldsymbol{x}, \boldsymbol{z} - \boldsymbol{x}\rangle = \langle \boldsymbol{y}' - \boldsymbol{x}, \boldsymbol{z} - \boldsymbol{x}\rangle > 0$. □

Corollary 10. $k'(d) \ge \lceil d/2 \rceil$.

Proof. Let $m := \lceil d/2 \rceil$. Let $\boldsymbol{p}_1, \dots, \boldsymbol{p}_m$ be the vertices of a regular simplex in $\mathbb{R}^{m-1}$ inscribed in the unit sphere. Then the $\boldsymbol{p}_i$ and $\boldsymbol{u}_i := -\boldsymbol{p}_i$ satisfy the conditions of Theorem 9. It follows that $k'(d) \ge k'(2m-1) \ge m$. □

Theorem 11. *There exist $m := \lfloor \frac{1}{4} e^{d/20} \rfloor$ distinct points $\boldsymbol{p}_1, \dots, \boldsymbol{p}_m \in \mathbb{R}^d$ and unit vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_m \in \mathbb{R}^d$ such that for all distinct $1 \le i,j,k \le m$, the angle $\angle \boldsymbol{p}_i\boldsymbol{p}_j\boldsymbol{p}_k$ is acute, and condition* (17) *is satisfied.*

The proof of Theorem 11 is probabilistic, and is a modification of an argument of Erdős and Füredi [2]. Write $[d]$ for the set $\{1, 2, \ldots, d\}$ of all integers from 1 to d. For any $A \subseteq [d]$, let $\boldsymbol{\chi}(A) \in \{0,1\}^d$ denote its characteristic vector. The routine proofs of the following three lemmas are omitted.

Lemma 12 ([2, Lemma 2.3]). *Let A, B, and C be distinct subsets of $[d]$. Then we have $\angle\boldsymbol{\chi}(A)\boldsymbol{\chi}(C)\boldsymbol{\chi}(B) \leq \pi/2$, and equality holds iff $A \cap B \subseteq C \subseteq A \cup B$.*

Lemma 13 ([2]). *If A, B, and C are subsets of $[d]$ chosen independently and uniformly, then we have $\Pr\Big[A \cap B \subseteq C \subseteq A \cup B\Big] = (3/4)^d$.*

Lemma 14. *Let $A, B, C \subseteq [d]$ and consider the unit vector*

$$\boldsymbol{u} := (1/\sqrt{d})(\boldsymbol{\chi}([d]) - 2\boldsymbol{\chi}(A)).$$

Then we have $\langle \boldsymbol{u}, \boldsymbol{\chi}(A)\rangle \leq \langle \boldsymbol{u}, \boldsymbol{\chi}(B)\rangle$, with equality if and only if $A = B$. Also,

$$\langle \boldsymbol{u}, \boldsymbol{\chi}(B) - \boldsymbol{\chi}(C)\rangle \geq \langle \boldsymbol{u}, \boldsymbol{\chi}(C) - \boldsymbol{\chi}(A)\rangle$$

if and only if

$$4\,|A \cap C| + |B| \geq 2\,|A \cap B| + |A| + 2\,|C|\,.$$

Lemma 15. *If A, B, and C are subsets of $[d]$ chosen independently and uniformly, then we have*

$$\Pr\Big[4\,|A \cap C| + |B| \geq 2\,|A \cap B| + |A| + 2\,|C|\Big] \leq \left(\frac{65}{72}\right)^d < e^{-d/10}.$$

Proof. Let X be the random variable

$$X := 4\,|A \cap C| + |B| - 2\,|A \cap B| - |A| - 2\,|C| = \sum_{i=1}^{d} X_i,$$

where X_i is the contribution of the element $i \in [d]$ to X, that is,

$$X_i := \begin{cases} 1 & \text{if } i \in B \setminus (A \cup C) \text{ or } i \in (A \cap C) \setminus B, \\ 0 & \text{if } i \in A \cap B \cap C \text{ or } i \notin A \cup B \cup C, \\ -1 & \text{if } i \in A \setminus (B \cup C) \text{ or } i \in (B \cap C) \setminus A, \\ -2 & \text{if } i \in C \setminus (A \cup B) \text{ or } i \in (A \cap B) \setminus C. \end{cases}$$

Note that

$$\Pr[X_i = 1] = \Pr[X_i = 0] = \Pr[X_i = -1] = \Pr[X_i = -2] = 1/4.$$

We now bound $\Pr[X \geq 0]$ from above. For any $\lambda \geq 1$,

$$\begin{aligned}\Pr[X \geq 0] &= \Pr\left[\lambda^X \geq 1\right] \\ &\leq \mathrm{E}\left[\lambda^X\right] = \prod_{i=1}^{d} \mathrm{E}\left[\lambda^{X_i}\right] = \left(\frac{\lambda + 1 + \lambda^{-1} + \lambda^{-2}}{4}\right)^d,\end{aligned}$$

where we used Markov's inequality and independence. Set $\lambda := 3/2$, which is close to minimizing the right-hand side. This gives $\Pr[X \geq 0] \leq (65/72)^d$. □

Proof of Theorem 11. Let $m := \lfloor (1/4)e^{d/20} \rfloor$. Choose subsets $A_1, \ldots, A_{2m}$ randomly and independently from the set $[d]$. For $i \in [d]$, define $\boldsymbol{p}_i := \boldsymbol{\chi}(A_i)$ and $\boldsymbol{u}_i := (1/\sqrt{d})(\boldsymbol{\chi}([d]) - 2\boldsymbol{\chi}(A_i))$. Let $i, j, k \in [d]$ be distinct.

Assume that A_i, A_j, A_k are distinct sets. Then by Lemma 12, $\angle \boldsymbol{p}_i\boldsymbol{p}_k\boldsymbol{p}_j$ fails to be acute if and only if

$$A_i \cap A_j \subseteq A_k \subseteq A_i \cup A_j, \tag{18}$$

and condition (17) is violated if and only if

$$\langle \boldsymbol{u}_i, \boldsymbol{\chi}(A_i) - \boldsymbol{\chi}(A_j)\rangle \geq \langle \boldsymbol{u}_i, \boldsymbol{\chi}(A_k) - \boldsymbol{\chi}(A_j)\rangle \tag{19}$$

or

$$\langle \boldsymbol{u}_i, \boldsymbol{\chi}(A_k) - \boldsymbol{\chi}(A_j)\rangle \geq \langle \boldsymbol{u}_i, \boldsymbol{\chi}(A_j) - \boldsymbol{\chi}(A_i)\rangle. \tag{20}$$

Condition (19) is equivalent to $\langle \boldsymbol{u}_i, \boldsymbol{\chi}(A_i)\rangle \geq \langle \boldsymbol{u}_i, \boldsymbol{\chi}(A_k)\rangle$. This, in turn, is equivalent to $A_i = A_k$, by the first statement of Lemma 14, contradicting our assumption that A_i, A_j, A_k are distinct. By the second statement of Lemma 14, (20) is equivalent to

$$4|A_i \cap A_j| + |A_k| \geq 2|A_i \cap A_k| + |A_i| + 2|A_j|. \tag{21}$$

Thus, for distinct points $\boldsymbol{p}_i$, $\boldsymbol{p}_j$, $\boldsymbol{p}_k$, at least one of the conditions (18) and (21) holds if and only if $\angle \boldsymbol{p}_i\boldsymbol{p}_k\boldsymbol{p}_j$ is a right angle or condition (17) is violated.

Note that if some two of the sets coincide, say $A_i = A_k$, then (18) also holds. Let us call a triple of distinct numbers (i, j, k) *bad* if at least one of (18) and (21) holds. It follows that if no triple (i, j, k) is bad, then all points $\boldsymbol{p}_i$ are distinct, all angles $\angle \boldsymbol{p}_i\boldsymbol{p}_j\boldsymbol{p}_k$ are acute, and condition (17) is also satisfied. We will show that with positive probability, some m of the $A_1, \ldots, A_{2m}$ will be without bad triples, which will prove the theorem.

By Lemmas 13 and 15 and the union bound, we obtain that

$$\Pr\Big[(i, j, k) \text{ is bad}\Big] \leq (3/4)^d + e^{-d/10} < 2e^{-d/10}.$$

By linearity of expectation, the expected number of bad triples is at most

$$2m(2m-1)(2m-2)2e^{-d/10} < 16m^3e^{-d/10}.$$

In particular, there exists a choice of subsets $A_1, \dots, A_{2m} \subseteq [d]$ with less than $16m^3e^{-d/10}$ bad triples. For each bad triple (i, j, k), remove A_i from $\{A_1, \dots, A_{2m}\}$. We are left with more than $2m - 16m^3e^{-d/10}$ sets without any bad triple. Since $m \leq (1/4)e^{d/20}$ implies that $2m - 16m^3e^{-d/10} \geq m$, we obtain m points $\boldsymbol{p}_i$ with unit vectors $\boldsymbol{u}_i$ satisfying the theorem. □

Corollary 16. *$k'(d) \geq d - O(\log d)$.*

Proof. Let n be the unique integer such that

$$\lfloor (1/4)e^{n/20} \rfloor + n \leq d < \lfloor (1/4)e^{(n+1)/20} \rfloor + n + 1.$$

By Theorems 11 and 9, $k'(m+n+1) \geq m$ for any $m = 2, \dots, \lfloor (1/4)e^{(n+1)/20} \rfloor$. In particular, if we take $m = d - n - 1$, we obtain

$$k'(d) \geq d - n - 1 > d - 20\log(4d) - 1. \qquad \square$$

Acknowledgement

We thank Endre Makai for a careful reading of the manuscript and for many enlightening comments.

References

[1] P. Erdős, *On sets of distances of n points in Euclidean space*, Magyar Tud. Akad. Mat. Kut. Int. Közl. **5** (1960), 165–169.

[2] P. Erdős and Z. Füredi, *The greatest angle among n points in the d-dimensional Euclidean space*, North-Holland Math. Stud. **75**, North-Holland, Amsterdam, 1983. pp. 275–283.

[3] P. Erdős and A. H. Stone, *On the structure of linear graphs*, Bull. Amer. Math. Soc. **52** (1946), 1087–1091.

[4] B. Grünbaum, *A proof of Vázsonyi's conjecture*, Bull. Research Council Israel, Section A **6** (1956), 77–78.

[5] A. Heppes, *Beweis einer Vermutung von A. Vázsonyi*, Acta Math. Acad. Sci. Hungar. **7** (1956), 463–466.

[6] A. Kupavskii, *Diameter graphs in* $\mathbb{R}^4$, arXiv:1306.3910.

[7] E. Makai, Jr. and H. Martini, *On the number of antipodal or strictly antipodal pairs of points in finite subsets of* $\mathbb{R}^d$, Applied geometry and discrete mathematics, 457–470, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 4, Amer. Math. Soc., Providence, RI, 1991.

[8] E. Makai, Jr. and H. Martini, *On the number of antipodal or strictly antipodal pairs of points in finite subsets of* $\mathbb{R}^d$*. II*, Period. Math. Hungar. **27** (1993) 185–198.

[9] E. Makai, Jr., H. Martini, H. Nguên, V. Soltan, and I. Talata, *On the number of antipodal or strictly antipodal pairs of points in finite subsets of* $\mathbb{R}^d$*. III*, manuscript.

[10] H. Martini and V. Soltan, *Antipodality properties of finite sets in Euclidean space*, Discrete Math. **290** (2005), 221–228.

[11] J. Pach, *A remark on transversal numbers*, In: The mathematics of Paul Erdős II, eds. R. L. Graham et al., Algorithms and Combinatorics, 14, Springer, Berlin, 1997. pp. 310–317.

[12] J. Pach and K. J. Swanepoel, *Double-normal pairs in the plane*, manuscript.

[13] S. Straszewicz, *Sur un problème géométrique de P. Erdős*, Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 39–40, IV–V.

[14] K. J. Swanepoel, *Unit distances and diameters in Euclidean spaces*, Discrete Comput. Geom. **41** (2009), 1–27.