Conflict-free colorings of graphs

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Abstract

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1 Introduction

Let H be a hypergraph with vertex set $V(H)$ and (hyper)edge set $E(H)$, and let $c : V(H) \to \{1, 2, 3, \ldots\}$ be a coloring of its vertex set. We say that c is a *proper coloring* if no edge $E \in E(H)$ consisting of at least two points is monochromatic. The smallest number of colors required for such a coloring is called the *chromatic number* of H, and is denoted by $\chi(H)$. A coloring is a *rainbow coloring* if for every edge $E \in E(H)$, no two vertices of E receive the same color. The minimum number of colors, $\chi_{\mathrm{RB}}(H)$, used in a rainbow coloring is the *rainbow chromatic number* of H. Motivated by a frequency assignment problem for cellular networks, Even, Lotker, Ron, and Smorodinsky [ELRS03] introduced an intermediate notion: a coloring of H is called *conflict-free* if every non-empty edge $E \in E(H)$ contains a vertex whose color does not get repeated in E. The minimum number of colors in such a coloring is the *conflict-free chromatic number*, denoted by $\chi_{\mathrm{CF}}(H)$. Obviously, every rainbow coloring is conflict-free and every conflict-free coloring is proper, therefore we have

$$\chi(H) \le \chi_{\mathrm{CF}}(H) \le \chi_{\mathrm{RB}}(H),$$

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for every hypergraph H.

For graphs (2-uniform hypergraphs), the above three notions coincide. However, they are not necessarily the same for hypergraphs formed by the neighborhoods of the vertices. Given a graph G and a vertex $x \in V(G)$, the *neighborhood* $N_G(x) = N(x)$ of x is defined as the set consisting of x and all vertices in G connected to x. The set $\dot{N}_G(x) = \dot{N}(x) = N(x) \setminus \{x\}$ is called the *pointed neighborhood* of x. Depending on which notion we use, we arrive at the definition of further interesting graph parameters. The *chromatic parameter* $\kappa(G)$, the *conflict-free chromatic parameter* $\kappa_{\mathrm{CF}}(G)$, and the *rainbow chromatic parameter* $\kappa_{\mathrm{RB}}(G)$ of G are defined as $\chi(H)$, $\chi_{\mathrm{CF}}(H)$, and $\chi_{\mathrm{RB}}(H)$ for the hypergraph H with $V(H) = V(G)$, $E(H) = \{N_G(x) : x \in V(G)\}$. The *pointed* versions of these parameters, $\dot{\kappa}(G)$, $\dot{\kappa}_{\mathrm{CF}}(G)$, and $\dot{\kappa}_{\mathrm{RB}}(G)$ are defined analogously, except that instead of H we have to consider the hypergraph $\dot{H}$ with edge set $E(\dot{H}) = \{\dot{N}_G(x) : x \in V(G)\}$. According to the definitions, for every graph G, we have

$$\kappa(G) \leq \kappa_{\mathrm{CF}}(G) \leq \kappa_{\mathrm{RB}}(G),$$

$$\dot{\kappa}(G) \leq \dot{\kappa}_{\mathrm{CF}}(G) \leq \dot{\kappa}_{\mathrm{RB}}(G).$$

Let us start with an example. Let K'_n be the graph obtained from the complete graph K_n on n vertices by subdividing each edge with a new vertex. Each pair from the n original vertices form the pointed neighborhood of a new vertex, so they should all get different colors at a proper coloring of the corresponding hypergraph $\dot{H}$. It is not hard to find a rainbow coloring of this hypergraph with n colors showing

$$\dot{\kappa}(K'_n) = \dot{\kappa}_{\mathrm{CF}}(K'_n) = \dot{\kappa}_{\mathrm{RB}}(K'_n) = n.$$

On the other hand K'_n is bipartite and a proper coloring of the graph is always a conflict-free coloring of the hypergraph formed by the neighborhoods. This shows that for $n \geq 2$ we have

$$\kappa(K'_n) = \kappa_{\mathrm{CF}}(K'_n) = 2.$$

Finally it is not hard to show that $\kappa_{\mathrm{RB}}(K'_n) = n$ for odd n and $\kappa_{\mathrm{RB}}(K'_n) = n + 1$ for even n.

It is also clear that $\kappa(G) \leq \dot{\kappa}(G)$, provided that G has at least one vertex of degree larger than *one*. Indeed, consider a coloring of the vertices (with at least *two* colors) such that no pointed neighborhood of size larger than *one* is monochromatic. This means that the (non-pointed) neighborhood of a vertex can only be monochromatic for vertices of degree at most one.

One can then fix the potential problems at degree one vertices by simply recoloring the offending vertex. Note that these two parameters can be arbitrarily far apart as the graphs K'_n show.

The parameters $\kappa_{\mathrm{RB}}(G)$ and $\dot\kappa_{\mathrm{RB}}(G)$ are easy to express as standard chromatic numbers. We have $\dot\kappa_{\mathrm{RB}}(G) = \chi(G^2)$, where G^2 is the graph on the vertex set of G with edges representing paths of length two. Similarly $\kappa_{\mathrm{RB}}(G)$ is the chromatic number of the graph obtained from G^2 by adding the edges of G. For the relation between the two parameter we have

$$\dot\kappa_{\mathrm{RB}}(G) \leq \kappa_{\mathrm{RB}}(G) \leq 2\dot\kappa_{\mathrm{RB}}(G).$$

Here the first inequality is trivial, the second comes from the fact that in a rainbow coloring of the pointed neighborhoods the colorclasses span matchings.

There is no direct inequality between the conflict-free chromatic parameter and its pointed variant. For the triangle K_3, we have $\kappa_{\mathrm{CF}}(K_3) = 2$, while $\dot\kappa_{\mathrm{CF}}(K_3) = 3$. In contrast, consider the graph G obtained from the complete graph K_4 by subdividing a single edge with a vertex. It is easy to check that $\dot\kappa_{\mathrm{CF}}(G) = 2$, but $\kappa_{\mathrm{CF}}(G) = 3$. As the graph K'_n show no function of $\kappa_{\mathrm{CF}}(G)$ bounds $\dot\kappa_{\mathrm{CF}}(G)$ in general, but in the other direction we have

$$\kappa_{\mathrm{CF}}(G) \leq 2\dot\kappa_{\mathrm{CF}}(G).$$

NEM ÉLES This inequality holds because in a conflict free coloring of the pointed neighborhoods all neighborhoods $N(x)$ also have a vertex whose color is not repeated in $N(x)$ unless x has degree one in the subgraph spanned by one of the color classes. One can fix these offending neighborhoods by carefully spliting the color classes in two.

The problem of bounding the conflict-free chromatic parameters for *visibility graphs* of systems of points or of other geometric figures was raised by P. Cheilaris and A. Holmsen. In the present note, we consider their behavior for general graphs.

The graph K'_m has $\binom{n=(m}{2)+m}$ vertices and $\dot\kappa_{\mathrm{CF}}(K'_m) = m = \Theta(\sqrt{n})$. This is almost maximal among graphs of n vertices and even among hypergraphs with n edges:

Theorem 1 *The conflict-free chromatic number of a hypergraph is $O(\sqrt{n})$, where n is the number of edges.*

In case we have a lower bound on the size of the edges we can prove more:

Theorem 2 *The conflict free chromatic number of of a hypergraph with n edges, each of which is of size at least $2t-1$ is $O(tn^{1/t}\log n)$.*

This result implies that the pointed conflict free chromatic parameters of n vertex graphs with minimum degree at least 5 is $O(n^{1/3}\log n)$. It is an interesting open problem if the degree bound in this last statement can be reduced to 3.

Somewhat surprisingly, the (*non-pointed*) conflict-free chromatic parameter $\kappa(G)$ of any graph G with n vertices is only polylogarithmic in n.

Theorem 3 *The conflict-free chromatic parameter of all graphs with n vertices is $O(\log^3 n)$.*

Next we show that the last bound is nearly optimal: there exist graphs of n vertices with conflict-free chromatic parameter $\Omega(\log n)$.

A graph G is called k-super-universal for some parameter $k \geq 1$ if for any set of vertices $A \subseteq V(G)$ with $|A| \leq k$ and for any $B \subseteq A$, there is a vertex $x \in V(G)$, $x \notin A$, which is connected to no element of B, but to all elements of $A \setminus B$.

We claim that if a graph G is k-super-universal, then $\kappa_{\text{CF}}(G) > k/2$. Indeed, let us color the vertices of G with at most $k/2$ colors, and we show that some neighborhood has all its colors repeated. Let B be the set of all vertices x that have a "unique" color, that is, a color not given to any vertex other than x. Further, let A be the set obtained from B by adding two representative vertices for each "non-unique" color. Clearly, $|A| \leq k$ and by the super-universality G has a vertex x not in A that has no neighbor in B and having all vertices in $A \setminus B$ as neighbors. Clearly, every color is repeated in $N(x)$.

To show the existence of super-universal graphs we turn to random graphs. Let $G = G(n, 1/2)$ be the random graph on n vertices with edge probability $1/2$. It is well known (as simple to show) that G is almost surely *k-super-universal* for some $k = \Omega(\log n)$ (in fact $k = \log n - O(\log\log n)$ for the binary logarithm functiun). This establishes the existence of n-vertex graphs G with $\kappa_{\text{CF}}(G) = \Omega(\log n)$.

2 Proof of Theorems 1 and 2

Consider an arbitrary set V of m vertices and color each vertex independently according the geometric distribution with parameter p. That is, we assign color $i = 1, 2, \ldots$ to a vertex with probability $p(1-p)^{i-1}$. It will be more convenient to consider this process in the following way. We take the vertices in V in a preassigned order one by one and for each vertex independently with probability p we assign the color 1. Next, again in order and independently for each vertex not colored, we assign it color 2, etc. This way we assign the colors to the vertices one by one and at any point in time, the distribution of the partition induced by the coloring of the set V' of yet uncolored vertices is the same as in the procedure when we color the set V' only.

We claim that the probability of V receiving at most $m/2$ colors at most $(3mp)^{\lceil m/2 \rceil}$. For the proof of this claim let σ be a partition of V into k nonempty classes. The probability of obtaining σ as the partition induced by the coloring is at most p^{m-k} as in the above procedure we have to choose coloring at the right moment for the $m-k$ vertices that are not first in their equivalence class. Thus the probability of using exactly k colors is at most $X_k = k^m p^{m-k}/k!$, since the total number of k-partitions of m elements is $S(m,k) < k^m/k!$. As $p < 1/m$ (or the claimed bound is meaningless) the sequence X_k is exponentially increasing for $k \le m/2$. The total probability of using at most $m/2$ colors is at most $\sum_{k=1}^{\lfloor m/2 \rfloor} X_k < 2X_{\lfloor m/2 \rfloor}$ that can easily be bounded by $(3mp)^{\lceil m/2 \rceil}$.

Clearly, if no color is unique in V, then there are at most $m/2$ colors, so the above bound applies. In case m is large, the above bound is meaningless, so we use the following observation instead. Let $m' < m$ be arbitrary. If there is no unique color in V, then there is no unique color among the set V' of the m' vertices that got colored last, or if there is, then there is no unique color among the last $m' - 1$ vertices.

Using these bounds the proof of Theorem 2 is straightforward. We apply the above coloring procedure for the vertex set with the parameter $p = n^{-1/t}/(12t)$. The probability that an edge does not have a unique color is at most $(6pt)^t < 1/(4n)$ for an edge of size $2t-1$ or $2t$ by the above formula. For larger edges we use the observation that not having a unique color implies that either the last $2t$ or the last $2t-1$ vertices in the edge have no unique color. Thus no edge avoids a unique color with probability over $1/(2n)$. Notice that if $|V| = O(n^2)$, then with high probability we use $O(tn^{1/t} \log n)$ colors. This shows the existence of a conflict free coloring with this many colors.

Finally to handle the $|V| > 2n^2$ case notice that $t \le n$ can be assumed as for larger t the bound becomes worse. Now select $2t-1$ vertices from each edge of the hypergraph. Let us restrict the hypergraph to the selected vertices, that is remove all vertices from the edges that are not selected for any edge. Let us find a conflict free coloring of this hypergraph and extend it by adding a new color for all the removed vertices. Clearly, the coloring so obtained is also conflict free. This proves Theorem 2.

To prove Theorem 1 we apply (the proof of) the $t = 3$ case of Theorem 2. The randomized coloring there makes suer that there are no problems with edges of size at least 5. We apply deterministic techniques to ensure that smaller edges receive a rainbow coloring. Let V be the vertex set of the n edge hypergraph we want to color. We assume without loss of generality that $|V| \leq 2n$. As in the previous proof, if this assumption fails we keep only 2 vertex per edge and find a conflict free coloring of the restricted hypergraph, then add a new color for the rest of the vertices.

Let G be a graph on the vertex set V in which two vertices are connected if they are in a common edge of the hypergraph of size at most 4. We order the vertices in V in non-increasing order of their degree in G. We color them with a modified version of the randomized procedure described at the beginning of the section. In round i we take the vertices in V that has not received any color in their fixed order and one by one we decide if we color them with color i. This time there is no complete independence, however. We ensure that we obtain a proper coloring of H. If we have previously colored a neighbor of a vertex v to color i we skip vertex v and do not color it in this round. However if no neighbor of v received color i so far, then we give color i to v with probability $p = n^{-1/3}/36$.

Clearly, small edges of the hypergraph, upto size 4 receive reinbow coloring with probability one. The probability that a larger edge has no unique color can be bounded in the same way as in the proof of Theorem 2, so that probability is at most $1/(2n)$. Finally we need to bound the number of colors used in the process. With high probability every vertex will be colored within $O(n^{1/3} \log n)$ rounds in which coloring is not forbidden by an edge of G that connects it to a vertex earlier in the order. If vertex v is connected to k earlier vertices, then these vertices must all be at least degree k, so G must have at least $k^2/2$ edges. As G has at most $6n$ edges we have $k = O(\sqrt{n})$ and Theorem 1 is proved.

3 Proof of Theorem 2

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