Seminar on Combinatorial Computing
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Room 8404, Graduate Center
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New proof of a theorem on recurrence relations

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Abstract

The following theorem relating rational generating functions and homogeneous recurrence relations and their solutions is a basic result in combinatorial analysis.

Theorem. Suppose that $d \geq 1$ is a positive integer, and suppose that $\alpha_1, \ldots, \alpha_d$ are d complex numbers with $\alpha_d \neq 0$. The following conditions on a function $f : \mathbf{N} \to \mathbf{C}$ are equivalent:

i)

$$\sum_{n \geq 0} f(n) x^n = \frac{P(x)}{Q(x)},$$

where $Q(x) = 1 + \alpha_1 x + \cdots + \alpha_d x^d$ and $P(x)$ is a complex polynomial in x of degree less than d.

ii) For each $n \geq 0$,

$$f(n+d) + \alpha_1 f(n+d-1) + \cdots + \alpha_d f(n) = 0.$$

iii) For each $n \geq 0$,

$$f(n) = \sum_{i=1}^{k} P_i(n) \gamma_i^n,$$

where $1 + \alpha_1 x + \cdots + \alpha_d x^d = \prod_{i=1}^{k} (1 - \gamma_i x)^{d_i}$, the γ_i's are distinct, and $P_i(n)$ is a polynomial in n of degree less than d_i.

The statement was taken from Richard Stanley's Combinatorics and Commutative Algebra. We give a simpler proof than the one sketched there.