Geometric Graph Theory Problems & Exercises
UNT Denton, May 28–June 1, 2002

1. Prove that $K_{3,3}$ cannot be drawn in the plane so that any two non-adjacent edges meet an even number of times.

2. For every n, construct a planar graph with n vertices, whose every straight-line drawing in the plane has aspect ratio at least cn. (The *aspect ratio* of a drawing is the ratio of the lengths of the longest and shortest edges; $c > 0$ is a suitable constant.)

3. **(a)** Construct a planar graph G whose vertices cannot be represented by non-crossing straight-line segments in the plane so that two segments are incident if and only if the corresponding vertices of G are adjacent.

(b) (Open problem) Can one represent the vertices of every 3-colorable planar graph G by straight-line segments in the plane so that two segments cross each other if and only if the corresponding vertices of G are adjacent?

4. (Berlekamp) Let $F_1, F_2, \ldots, F_m$ be subsets of an n element set such that (a) $|F_i|$ is odd, for every i; (b) $|F_i \cap F_j|$ is even for every $i \neq j$.
Prove that $m \leq n$.

5. Given a system of disks $D_1, D_2, \ldots, D_n$ in the plane, connect any pair of their centers by a segment if and only if their boundaries have at least one intersection point which lies on the boundary of $\cup_{i+1}^{n} D_i$.
Prove that there is no crossing in the resulting drawing.

6. Construct a set of n points 3-space, in which the diameter occurs between at least $2n - 2$ different pairs of points. (*diameter* = maximal distance)

7. Construct geometric graphs with n vertices and $\frac{5}{2}n + O(1)$ edges, containing no 3 pairwise disjoint edges. (Open problem: can you beat this? The best known upper bound is $3n$.)

8. (Open problem) Does there exist a constant $c > 0$ such that for any 2-coloring of the edges of a complete geometric graph with n vertices, one can find a non-crossing monochromatic path of length at least cn?

9. For every even $n > 6$, construct a complete geometric graph with n vertices, which has no $\frac{n}{2} - 1$ pairwise crossing edges.

10. (Davenport-Schinzel) Determine the maximum length of a sequence whose elements are from $\{1, 2, \ldots, n\}$, no 2 consecutive elements are the same, and there is no alternating subsequence of length 4:

$$\ldots a \ldots b \ldots a \ldots b \ldots \quad (a \neq b)$$

11. (Open problem) Let $e_3(n)$ denote the maximum number of edges in a drawing of an n vertex graph, in which every edge participates in at most 3 crossings. It is known that $e_3(n) \geq 5.5n + o(n)$. Is this bound tight? (The best recent upper bound is 17n/3.)

12. Show that if $odd\!-\!cr(G) = 1$, then $lin\!-\!cr(G) = 1$. What happens if $odd\!-\!cr(G) = 2$?

13. Prove that for any C_6-free graph G with n vertices and $e \geq 4n$ edges, we have $cr(G) \geq ce^5/n^4$, where $c > 0$ is a suitable constant.

14. Prove that if $pair - cr(G) = k$ for some G, then $cr(G) \leq k(2k-1)$.

15. Prove that the maximum number of crossing-free edges in a complete geometric graph with $n \geq 4$ vertices is $2n-2$. (An edge is *crossing-free*, if it does not cross any other edge.)

16. **(a)** Let $\Delta(n)$ be the maximum number of triangles induced by a set of n points in general position in 3-space that can be chosen so that there are no 2 disjoint triangles.

Prove that $\Delta(n) = \Theta(n^2)$.

(b) Let $\Delta^*(n)$ be the maximum number of triangles induced by a set of n points in general position in 3-space that can be chosen so that no 2 non-adjacent triangles cross. Estimate $\Delta^*(n)$.

Enjoy your stay! Good luck with the open problems and the exercises! - J.P.