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COMPUTING LIMIT LOADS BY MINIMIZING A SUM OF NORMS*

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Abstract. This paper treats the problem of computing the collapse state in limit analysis for a solid with a quadratic yield condition, such as, for example, the von Mises condition. After discretization with the finite element method, using divergence-free elements for the plastic flow, the kinematic formulation reduces to the problem of minimizing a sum of Euclidean vector norms, subject to a single linear constraint. This is a nonsmooth minimization problem, since many of the norms in the sum may vanish at the optimal point. Recently an efficient solution algorithm has been developed for this particular convex optimization problem in large sparse form.

The approach is applied to test problems in limit analysis in two different plane models: plane strain and plates. In the first case more than 80% of the terms in the objective function are zero in the optimal solution, causing extreme ill conditioning. In the second case all terms are nonzero. In both cases the method works very well, and problems are solved which are larger by at least an order of magnitude than previously reported. The relative accuracy for the solution of the discrete problems, measured by duality gap and feasibility, is typically of the order 10^{-8}.

Key words. limit analysis, plasticity, finite element method, nonsmooth optimization

AMS subject classifications. 65K10, 65N30, 73E20, 90C06, 90C25, 90C90

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1. Introduction.

The problem of limit analysis is the following: given a load distribution on a rigid plastic solid, what is the maximum multiple of this load that the solid can sustain without collapsing? And when collapse does occur, what are the fields of stresses and plastic flow in the collapse state? In particular, it is of interest to find the *plastified region*, where the stresses are at the yield surface and where plastic deformation takes place. It is harder to find the collapse fields than the collapse multiplier. In typical cases these fields are not uniquely determined, in contrast to the equilibrium problems within the elastic model.

We shall use the following notation:

V	:	domain occupied by the solid
S	:	fixed part of V's surface
T	:	free and possibly loaded part of the surface
$\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x})$	:	volume force at $\boldsymbol{x} \in V$
$\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{x})$	:	surface force at $\boldsymbol{x} \in T$
$\boldsymbol{\sigma} = (\sigma_{ij})$	:	stress tensor (symmetric)
$\boldsymbol{\sigma} \in \mathcal{K}$	:	the yield condition
$\boldsymbol{u} = (u_i)$	:	plastic flow field
$\boldsymbol{\varepsilon} = (\varepsilon_{ij})$	:	plastic deformation rate tensor defined by

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$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The work rate for the pair of forces $(\boldsymbol{f}, \boldsymbol{g})$ and a virtual plastic flow $\boldsymbol{u}$ is

$$F(\boldsymbol{u}) = \int_V \boldsymbol{f} \cdot \boldsymbol{u} + \int_T \boldsymbol{g} \cdot \boldsymbol{u}. \tag{1.1}$$

The work rate for the internal stresses is given by the bilinear form

$$a(\boldsymbol{\sigma}, \boldsymbol{u}) = \int_V \sum_{i,j} \sigma_{ij}\varepsilon_{ij} d\boldsymbol{x} = \int_V \sum_{i,j} \sigma_{ij} \frac{\partial u_i}{\partial x_j} d\boldsymbol{x}. \tag{1.2}$$

We assume that the yield condition is of the form

$$\boldsymbol{\sigma} \in \mathcal{K} \Leftrightarrow K(\boldsymbol{\sigma}) \leq 1, \tag{1.3}$$

where K is a quadratic function in the components of $\boldsymbol{\sigma}$. For example, the von Mises condition is

$$(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) \leq 2\sigma_0^2, \tag{1.4}$$

σ_0 being the yield stress in simple tension. The yield condition must be satisfied at every point in the solid. Note that (1.4) does not bound the diagonal components of $\boldsymbol{\sigma}$.

The limit multiplier λ^* is given by (Christiansen [12, 16])

$$\begin{aligned}
\lambda^* &= \max\{\lambda \mid \exists\, \boldsymbol{\sigma} \in \mathcal{K} : a(\boldsymbol{\sigma}, \boldsymbol{u}) = \lambda F(\boldsymbol{u}) \quad \forall\, \boldsymbol{u}\} && (1.5)\\
&= \max_{\sigma \in \mathcal{K}} \min_{F(u)=1} a(\boldsymbol{\sigma}, \boldsymbol{u}) && (1.6)\\
&= \min_{F(u)=1} \max_{\sigma \in \mathcal{K}} a(\boldsymbol{\sigma}, \boldsymbol{u}) && (1.7)\\
&= \min_{F(u)=1} D(\boldsymbol{u}), && (1.8)
\end{aligned}$$

where

$$D(\boldsymbol{u}) = \max_{\sigma \in \mathcal{K}} a(\boldsymbol{\sigma}, \boldsymbol{u}). \tag{1.9}$$

The expression (1.5) states the existence of an admissible stress tensor $\boldsymbol{\sigma} \in \mathcal{K}$ which is in equilibrium with the external forces $(\lambda \boldsymbol{f}, \lambda \boldsymbol{g})$. Equation (1.6) follows from simple linear algebra, while (1.7) is the duality theorem of limit analysis proved in [14] and [16, section 5]. The expressions (1.5) and (1.8) are traditionally known, respectively, as the static and kinematic principles of limit analysis.

The solution to the problem of limit analysis consists of the triple $(\lambda^*, \boldsymbol{\sigma}^*, \boldsymbol{u}^*)$, where $(\boldsymbol{\sigma}^*, \boldsymbol{u}^*)$ is a saddle point for (1.6)–(1.7). The functions $\boldsymbol{\sigma}^*$ and $\boldsymbol{u}^*$ are then fields of stress and flow in the collapse state. It follows from the kinematic principle (1.8) that

$$\lambda^* = a(\boldsymbol{\sigma}^*, \boldsymbol{u}^*) = D(\boldsymbol{u}^*) = \max_{\sigma \in \mathcal{K}} a(\boldsymbol{\sigma}, \boldsymbol{u}^*).$$

Inserting the form (1.2) for $a(\boldsymbol{\sigma}, \boldsymbol{u})$ we get

$$\int_V \sum_{i,j} \sigma_{ij}^* \; \varepsilon_{ij}(\boldsymbol{u}^*) \; d\boldsymbol{x} = \max_{\sigma \in \mathcal{K}} \int_V \sum_{i,j} \sigma_{ij} \; \varepsilon_{ij}(\boldsymbol{u}^*) \; dx,$$

which leads to the principle of complementary slackness in limit analysis: at each point in the material where $\boldsymbol{\varepsilon}(\boldsymbol{u}^*)$ is nonzero, the collapse stress tensor $\boldsymbol{\sigma}^*$ must be at the yield surface, i.e., $K(\boldsymbol{\sigma}^*) = 1$. Furthermore, $\boldsymbol{\varepsilon}(\boldsymbol{u}^*)$ is an outward normal to the yield surface at $\boldsymbol{\sigma}^*$. Regions with $\boldsymbol{\varepsilon}(\boldsymbol{u}^*) = \mathbf{0}$ are rigid. Points where $\boldsymbol{\varepsilon}(\boldsymbol{u}^*) \neq \mathbf{0}$, implying that $\boldsymbol{\sigma}^*$ is at the yield surface, belong to the *plastified region.* Identification of these regions is an important part of the solution process.

The von Mises yield condition (1.4) is insensitive to the addition of any tensor of the form $\varphi \boldsymbol{I}$, where $\boldsymbol{I} = (\delta_{ij})$ is the unit tensor. This reflects the property that purely hydrostatic pressure (or underpressure) does not affect plastic collapse. We assume that the set $\mathcal{K}$ of admissible tensors in condition (1.3) either (a) is bounded (the easy case) or (b) has the property that

$$\boldsymbol{\sigma} \in \mathcal{K} \Leftrightarrow (\boldsymbol{\sigma} - \varphi \boldsymbol{I}) \in \mathcal{K} \quad \text{for any function} \quad \varphi. \tag{1.10}$$

The case where $\mathcal{K}$ is bounded occurs in the plane stress model and in the plate model. The unbounded case occurs in three-dimensional problems and in the plane strain model. We concentrate on the unbounded case, although we shall report computational results for the plate model as well.

Assume now that the yield condition satisfies (1.10). Then it is easy to see that the so-called energy dissipation rate $D(\boldsymbol{u})$ defined by (1.9) is finite if and only if $\boldsymbol{u}$ is divergence free, $\nabla \cdot \boldsymbol{u} = 0$; i.e., the plastic flow is incompressible. This condition is an infinite set of linear constraints on $\boldsymbol{u}$ in the minimization problem (1.8). For this reason the standard approach in limit analysis with unbounded yield set has been to solve the discrete form of the static principle (1.5). This problem is large, sparse, and ill-conditioned, partially due to the unbounded feasible set. Since efficient convex programming methods for such problems were not available, the quadratic yield condition was linearized, and the resulting linear program (LP) was solved with the simplex method (see, e.g., [2, 9, 12]).

In [17] and [18], it was demonstrated that interior-point methods are very effective for solving the LPs arising in limit analysis, making it possible to obtain solutions on finer grids. Furthermore it was noted that interior-point methods give more "physically correct" collapse fields than the simplex method. This is because in the typical case of nonunique or poorly determined solutions, the simplex method generates extreme-point solutions which may oscillate from node to node. By contrast, an interior-point method tends to generate solutions which are better centered in the optimal faces. A new and very efficient infeasible interior-point method for LPs [3] was applied to limit analysis in [5] (see also [16]). Problems one order of magnitude larger than before were solved.

In [15] and [16] further advances were made by abandoning the linearization of the quadratic yield condition, instead solving the discrete form of (1.5), which is a quadratically constrained optimization problem. This was done using the convex programming feature of MINOS [25]. The limit multiplier was more accurately determined than when using LP methods, but the collapse fields again displayed unphysical fluctuations due to the extreme point nature of the algorithm used by MINOS.

In the present paper, we report results obtained by direct minimization of the discrete analogue of (1.8), i.e., using the discrete kinematic principle. The objective function is convex but nondifferentiable. We solve this problem with the method developed in [4] and obtain results that are far more accurate than previously published.

The paper is organized as follows. In section 2, we introduce the discretization we use for limit analysis with an unbounded yield condition, approximating the plastic flow $\boldsymbol{u}$ and the stresses $\boldsymbol{\sigma}$ simultaneously by different finite element functions, for reasons which we shall explain. In section 3, we discuss the technique we use for solving the discrete optimization problem corresponding to (1.8). In section 4, we turn our attention to a plate bending problem with quite different characteristics but which nonetheless leads to a discrete optimization problem of the same type. In section 5 we give computational results.

2. The discretization. We assume that the yield condition is of the form (1.10) and impose the implicit constraint $\nabla \cdot \boldsymbol{u} = 0$. Then

$$a(\varphi \boldsymbol{I}, \boldsymbol{u}) = \int_V \varphi(\nabla \cdot \boldsymbol{u}) d\boldsymbol{x} = 0$$

for all scalar functions φ. Hence we need only consider stress tensors satisfying $\sum \sigma_{ii} = 0$ in the duality problem (1.5)–(1.8). With this restriction the set of admissible tensors is bounded, and hence the objective function $D(\boldsymbol{u})$ in the minimization problem (1.8) is finite for all $\boldsymbol{u}$ satisfying $\nabla \cdot \boldsymbol{u} = 0$. The details are given in [14] and in [16, section 5.4].

Using standard finite element spaces, it is straightforward to find a discrete representation for stresses satisfying $\sum \sigma_{ii} = 0$ and thus reduce the problem size. The constraint $\nabla \cdot \boldsymbol{u} = 0$ is a complication known from finite element computations in fluid mechanics, discussed in, e.g., Temam [28, sections 4.4–4.5]. The trick is to represent the flow $\boldsymbol{u}$ as a curl, $\boldsymbol{u} = \nabla \times \Psi$, which implies $\nabla \cdot \boldsymbol{u} = 0$. Instead of choosing finite elements for the flow $\boldsymbol{u}$ itself we discretize the vector Ψ to a finite element representation Ψ_h, such that the discrete flow $\boldsymbol{u}_h$ is given by

$$\boldsymbol{u}_h = \nabla \times \Psi_h. \tag{2.1}$$

(h is a linear measure of the element size in the discretization.) Care must be taken to ensure that $\boldsymbol{u}_h$ satisfies the boundary conditions.

This way we obtain not only a finite and computable objective function $D(\boldsymbol{u})$ for the minimization problem (1.8); we also get a reduction in problem size by removing compressible flow and purely hydrostatic pressure from the duality problem. There is a price to pay, though: the discrete flow $\boldsymbol{u}_h$ must be continuous because of the boundary condition and the derivatives in the expression (1.2). Hence $\boldsymbol{\Psi}_h$ must be of class C^1. In two space dimensions this implies the use of elements like the Argyris triangle or the Bell triangle. If the geometry permits a triangulation into rectangles (in two or three space dimensions), then the tensor product of the standard cubic C^1-elements (Bogner–Fox–Schmit rectangle) is a convenient choice. (All these finite elements are described in Ciarlet [21, section 2.2] and [22, section 9].)

In plane strain this approach is particularly attractive: $\boldsymbol{\Psi}_h$ only has one nonzero component which we shall denote Ψ_h, and (2.1) reduces to

$$\boldsymbol{u}_h = \left(\frac{\partial \Psi_h}{\partial x_2} \ , \ - \frac{\partial \Psi_h}{\partial x_1} \right). \tag{2.2}$$

The corresponding stress tensor of trace zero satisfies $\sigma_{22} = -\sigma_{11}$ and may be identified with the vector (σ_1, σ_2), where

$$\boldsymbol{\sigma}_h = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & -\sigma_1 \end{bmatrix}. \tag{2.3}$$

With this notation the von Mises yield condition in plane strain becomes

$$\sigma_1^2 + \sigma_2^2 \le \frac{1}{3}\sigma_0^2. \tag{2.4}$$

In order to be specific we briefly describe the discretization of the plane strain case with bicubic C^1-elements over rectangles (the Bogner–Fox–Schmit rectangle) for Ψ_h. The nodes for this finite element space are the vertices of the rectangles. There are four basis functions associated with each vertex μ, corresponding to the following nodal values of Ψ_h [22, p. 92]:

$$\Psi \ , \ \frac{\partial \Psi}{\partial x_1} \ , \ \frac{\partial \Psi}{\partial x_2} \ , \ \frac{\partial^2 \Psi}{\partial x_1 \partial x_2}.$$

According to (2.2) two of these nodal values correspond to the components of $\boldsymbol{u}_h$ at the vertex.

For the stress components σ_1 and σ_2 in (2.3) there are no boundary conditions imposed, and no derivatives occur in the expression (1.2) for $a(\boldsymbol{\sigma}, \boldsymbol{u})$, so the discrete stresses need not even be continuous. However, as we shall explain below, the dimension of the space of discrete stresses $\boldsymbol{\sigma}_h$ must satisfy a compatibility condition with the dimension of the space for Ψ_h which in this case has four degrees of freedom per vertex. The space of piecewise bilinear functions is too small, but the space of piecewise biquadratic element functions satisfies this condition. The nodes are the vertices of the rectangles, the midpoints of the sides, and the midpoint of the rectangles (see [22, p. 77]). Associated with each node N_ν there is a scalar basis function φ_ν characterized completely by being equal to one at this node and equal to zero at all other nodes. Consequently, there are the following basis functions for the space of discrete stresses $\boldsymbol{\sigma}_h$ given by (2.3):

$$\boldsymbol{\varphi}_\nu^1 = \begin{bmatrix} \varphi_\nu & 0 \\ 0 & -\varphi_\nu \end{bmatrix} \ , \quad \boldsymbol{\varphi}_\nu^2 = \begin{bmatrix} 0 & \varphi_\nu \\ \varphi_\nu & 0 \end{bmatrix}.$$

The discrete stress tensor may be written

$$\boldsymbol{\sigma}_h = \sum_\nu \left(\xi_\nu^1 \boldsymbol{\varphi}_\nu^1 + \xi_\nu^1 \boldsymbol{\varphi}_\nu^2 \right). \tag{2.5}$$

The yield condition (2.4) is imposed on the nodal values as follows:

$$\left(\xi_\nu^1\right)^2 + \left(\xi_\nu^2\right)^2 \le \frac{1}{3}\sigma_0^2 \quad \text{for all nodes} \quad \nu. \tag{2.6}$$

The expressions (1.1) for $F(\boldsymbol{u}_h)$ and (1.2) for $a(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$ may be expressed in the nodal values for $\boldsymbol{u}_h$ and $\boldsymbol{\sigma}_h$. For the details we refer to [6, pp. 7–8] (also to appear in [16, section 9]). With a suitable numbering of the nodal values, $\boldsymbol{x} = (x_n) \in \mathbb{R}^N$ for $\boldsymbol{\sigma}$ and $\boldsymbol{y} = (y_m) \in \mathbb{R}^M$ for $\boldsymbol{u}$, we get

$$F(\boldsymbol{u}_h) = \sum_{m=1}^{M} y_m b_m = \boldsymbol{b}^T \boldsymbol{y} \tag{2.7}$$

and

$$a(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) = \sum_{m=1}^{M} \sum_{n=1}^{N} y_m x_n a_{mn} = \boldsymbol{y}^T \boldsymbol{A} \boldsymbol{x} = \boldsymbol{x}^T \left(\boldsymbol{A}^T \boldsymbol{y} \right). \tag{2.8}$$

Let $\mathcal{K}_d$ denote the set of $\boldsymbol{x} \in \mathbb{R}^N$ for which the corresponding (ξ_ν^k) satisfies the yield condition (2.6), and define

$$D_d(\boldsymbol{y}) = \max_{x \in \mathcal{K}_d} \boldsymbol{x}^T(\boldsymbol{A}^T\boldsymbol{y}). \tag{2.9}$$

With this notation we get

$$\begin{aligned} \lambda_h^* &= \max\{\lambda \mid \exists\, \boldsymbol{x} \in \mathcal{K}_d : \boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{b}\} && (2.10)\\ &= \max_{x \in \mathcal{K}_d} \min_{b^T y = 1} \boldsymbol{y}^T \boldsymbol{A}\boldsymbol{x} \\ &= \min_{b^T y = 1} \max_{x \in \mathcal{K}_d} \boldsymbol{x}^T(\boldsymbol{A}^T\boldsymbol{y}) \\ &= \min_{b^T y = 1} D_d(\boldsymbol{y}). && (2.11) \end{aligned}$$

The duality between (2.10) and (2.11) follows immediately from [11, Theorem 2.1] but is standard in the finite-dimensional case.

From the discrete static form (2.10) it follows that $\lambda_h^* = 0$, if $\boldsymbol{b}$ does not belong to the range of $\boldsymbol{A}$, $R_A = \{\boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{R}^N\}$. In order to handle general external forces we must impose the consistency condition $R_A = \mathbb{R}^M$ or, in other words, the matrix $\boldsymbol{A}$ must have full row rank M. In particular, it is necessary that $N > M$. This is the compatibility condition on the space of discrete stresses $\boldsymbol{\sigma}_h$ mentioned above, and this is the reason why we must use piecewise biquadratic element functions for the components of $\boldsymbol{\sigma}_h$ instead of, e.g., piecewise bilinear elements.

The use of piecewise biquadratic elements for $\boldsymbol{\sigma}_h$ introduces an error in connection with the yield condition. The inequality (2.6) imposes the yield condition (2.4) for $\boldsymbol{\sigma}_h$ only at the nodes. With biquadratic element functions it may be violated between nodes. The maximum pointwise violation of the quadratic condition (2.4) may be as large as approximately 28% of the variation of a component of $\boldsymbol{\sigma}_h$ over the element. With a similar change of sign in the corresponding component of $\boldsymbol{\varepsilon}(\boldsymbol{u}_h)$ this may result in a value of (2.9) which is larger than the correct value $D(\boldsymbol{u}_h)$ defined by (1.9). In applications $\boldsymbol{\sigma}_h^*$ will approximate an exact $\boldsymbol{\sigma}^*$ which may have a jump discontinuity. In this case the total area of elements across which $\boldsymbol{\sigma}_h^*$ has maximum variation will decrease as $O(h)$ as h tends to zero, and the influence of the constraint violation on the discrete collapse multiplier λ_h^* can be considered part of the discretization error. In limit analysis we cannot expect faster convergence than $O(h)$ [12, 13, 16, 18]. We do not know to what extent this constraint violation actually contributes to the discretization error in λ_h^*.

3. Solution of the discrete problem. We approach the discrete problem in the kinematic form (2.11). There is only one linear equality constraint, but the objective function (2.9) is not differentiable. The matrix $\boldsymbol{A}$ is sparse with the usual finite element structure for problems in two space dimensions. Using the notation (2.5) for the x-variables we have

$$\boldsymbol{x}^T\left(\boldsymbol{A}^T\boldsymbol{y}\right) = \sum_\nu \sum_{k=1}^{2} \xi_\nu^k \left(\boldsymbol{A}^T\boldsymbol{y}\right)_\nu^k.$$

Substituting for $\boldsymbol{x} \in \mathcal{K}_d$ in (2.9) the normalized yield condition (2.6), i.e.,

$$\left(\xi_\nu^1\right)^2 + \left(\xi_\nu^2\right)^2 \le 1 \quad \text{for all nodes} \quad \nu,$$

we get

$$D_d(\boldsymbol{y}) = \sum_{\nu} \left(\left(\left(\boldsymbol{A}^T \boldsymbol{y} \right)^1_{\nu} \right)^2 + \left(\left(\boldsymbol{A}^T \boldsymbol{y} \right)^2_{\nu} \right)^2 \right)^{\frac{1}{2}},$$

a sum of Euclidean norms of two-dimensional vectors. Each term only involves two rows of $\boldsymbol{A}^T$, i.e., two columns of $\boldsymbol{A}$. These are precisely the two columns associated with the node ν, and they correspond to the value of the two components of $\boldsymbol{\sigma}_h$ at that node. Now let $\boldsymbol{A}_\nu$ denote the $M \times 2$ matrix consisting of these two columns, i.e., the two columns corresponding to the primal variables ξ^1_ν and ξ^2_ν. Then we get the following expression for the objective function (2.9):

$$D_d(\boldsymbol{y}) = \sum_{\nu} \left(\left(\boldsymbol{A}_\nu^T \boldsymbol{y} \right)^2_1 + \left(\boldsymbol{A}_\nu^T \boldsymbol{y} \right)^2_2 \right)^{\frac{1}{2}} = \sum_{\nu} \left\| \boldsymbol{A}_\nu^T \boldsymbol{y} \right\|. \tag{3.1}$$

The discrete problem (2.11) may now be written

$$\lambda_h^* = \min_{b^T y = 1} \sum_{\nu} \left\| \boldsymbol{A}_\nu^T \boldsymbol{y} \right\|, \tag{3.2}$$

where the sum is over the nodes for the discrete stresses as explained above. This is a problem of minimizing a sum of Euclidean norms subject to one linear constraint. In the optimal solution we typically expect a large number (in some applications more than 90%) of the terms in (3.2) to vanish and hence be nondifferentiable. The quantity $D_d(\boldsymbol{y})$ defined by (2.9) is the discrete analogue of $D(\boldsymbol{u})$ defined by (1.9). Hence the nodes for which $\boldsymbol{A}_\nu^T \boldsymbol{y} = \boldsymbol{0}$ correspond to points where $\boldsymbol{\epsilon}(\boldsymbol{u}) = \boldsymbol{0}$, i.e., to points in the rigid region where there is no local deformation.

Calamai and Conn [7, 8] and Overton [26] have developed second-order methods for problems like (3.2). The idea is to identify dynamically the zero-terms and then replace them by constraints of the type $\boldsymbol{A}_\nu^T \boldsymbol{y} = \boldsymbol{0}$. Specialized versions of Overton's method were applied to test problems in limit analysis in [27] (antiplane shear) and [23] (plane stress), respectively. In these applications, limited to bounded yield sets, only the plastic flow $\boldsymbol{u}$, and not the stresses, was discretized by finite elements, leading to a so-called lower bound method (see [12]). Our initial experiments minimizing (3.2) directly used the code of [26], but these were limited to coarse grids. The results presented in section 5 were obtained with a much more efficient algorithm developed by Andersen [4], extending the ideas of interior-point methods for LP to the sum-of-norms objective in (3.2). The interior-point method eliminates difficulties with degenerate solutions discussed in [7, 8, 23, 26, 27]. The algorithm is designed for unconstrained problems, but an efficient technique which is used to handle the single constraint $\boldsymbol{b}^T \boldsymbol{y} = 1$ is discussed in [4, Appendix A].

4. The plate problem. Limit analysis for plastic plates is described in [10]. The variables are the three components of the bending moments m_{11}, m_{22}, and $m_{12} = m_{21}$ and the transversal displacement rate u. The problem of limit analysis is formally the same as above with the following modifications:

$$\begin{aligned} a(\boldsymbol{m}, u) &= -\int_A \sum_{i,j=1}^{2} m_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \, da \\ &= \int_A \left(\frac{\partial u}{\partial x_1} \left(\frac{\partial m_{11}}{\partial x_1} + \frac{\partial m_{12}}{\partial x_2} \right) + \frac{\partial u}{\partial x_2} \left(\frac{\partial m_{12}}{\partial x_1} + \frac{\partial m_{22}}{\partial x_2} \right) \right) da \end{aligned} \tag{4.1}$$

$$= -\int_A u\left(\frac{\partial^2 m_{11}}{\partial x_1^2} + 2\frac{\partial^2 m_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 m_2}{\partial x_2^2}\right) da,$$

$$F(u) = \int_A f\ u\ da, \tag{4.2}$$

where A denotes the area of the plate and f is the transversal force. For computational purposes we always use the form (4.1) with one derivative on both $\boldsymbol{m}$ and u, so that standard finite element functions can be applied.

In the plate model the set of admissible moment tensors $\boldsymbol{m}$ is bounded. For example, the von Mises condition is in normalized form:

$$m_{11}^2 - m_{11}m_{22} + m_{22}^2 + 3m_{12}^2 \leq 1. \tag{4.3}$$

Consequently, $D(u)$ defined by

$$D(u) = \max_{m \in \mathcal{K}} a(\boldsymbol{m}, u) \tag{4.4}$$

is always finite.

With the substitutions (4.1), (4.2), and (4.3) the problem of limit analysis is formally the same as discussed above. The discretization is much simpler in the plate model because the energy dissipation rate $D(u)$ is finite for all u, so that standard finite element spaces may be used. As in [19] we use piecewise bilinear element functions for both u and the components of $\boldsymbol{m}$, but piecewise linear elements over triangles would do just as well. The nodes are the vertices of the rectangles, and the nodal values are the values of m_{11}, m_{22}, m_{12}, and u at the vertices.

The discrete moment tensor may be written

$$\boldsymbol{m}_h = \sum_\nu \left(\xi_\nu^{11}\boldsymbol{\varphi}_\nu^{11} + \xi_\nu^{22}\boldsymbol{\varphi}_\nu^{22} + \xi_\nu^{12}\boldsymbol{\varphi}_\nu^{12}\right), \tag{4.5}$$

where

$$\boldsymbol{\varphi}_\nu^{11} = \begin{bmatrix} \varphi_\nu & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{\varphi}_\nu^{22} = \begin{bmatrix} 0 & 0 \\ 0 & \varphi_\nu \end{bmatrix}, \quad \boldsymbol{\varphi}_\nu^{12} = \begin{bmatrix} 0 & \varphi_\nu \\ \varphi_\nu & 0 \end{bmatrix}; \tag{4.6}$$

φ_ν denotes the scalar bilinear element function equal to one at the node ν and zero at all other nodes. The yield condition is again imposed through the nodal values:

$$\left(\xi_\nu^{11}\right)^2 - \xi_\nu^{11}\xi_\nu^{22} + \left(\xi_\nu^{22}\right)^2 + 3\left(\xi_\nu^{12}\right)^2 \leq 1,$$

or equivalently,

$$\boldsymbol{\xi}_\nu^T \boldsymbol{Q} \boldsymbol{\xi}_\nu \leq 1 \quad \text{for all nodes } \nu, \tag{4.7}$$

where

$$\boldsymbol{Q} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \boldsymbol{\xi}_\nu = \begin{bmatrix} \xi_\nu^{11} \\ \xi_\nu^{22} \\ \xi_\nu^{12} \end{bmatrix}. \tag{4.8}$$

With piecewise bilinear elements the yield condition will be satisfied at every point if it is satisfied at the nodal points.

In analogy with (3.1) we get (for details we refer to [19])

$$D_d(\boldsymbol{y}) = \max_{\xi_\nu^T Q \xi_\nu \le 1} \sum_\nu \boldsymbol{\xi}_\nu^T \left(\boldsymbol{A}_\nu^T \boldsymbol{y} \right) = \sum_\nu \left\| \boldsymbol{C}^T \left(\boldsymbol{A}_\nu^T \boldsymbol{y} \right) \right\|. \tag{4.9}$$

$\boldsymbol{A}_\nu$ are now $M \times 3$ matrices and $\boldsymbol{C}$ is the Cholesky factor of Q^{-1} :

$$\boldsymbol{C} = \frac{1}{\sqrt{3}} \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The discrete problem for plates analogous to (3.2) is then

$$\lambda_h^* = \min_{b^T y = 1} \sum_\nu \left\| (\boldsymbol{A}_\nu \boldsymbol{C})^T \boldsymbol{y} \right\|. \tag{4.10}$$

This is again the minimization of a sum of norms subject to one linear constraint, and we use the same algorithm as in the previous section.

In general, quadratic yield conditions give rise to problems of the form (4.10). The $\boldsymbol{A}_\nu$ are columns of the equilibrium matrix $\boldsymbol{A}$, while $\boldsymbol{C}$ depends only on the yield condition.

5. Computational results. The new approach is applied to two collapse load problems in plane models: plane strain and plates. They give rise to two families of discrete optimization problems with different characteristics. The coarse mesh solutions, i.e., for small discrete problems, can be compared with known solutions and provide verification of the correctness of our implementation. The fine mesh solutions have not been computed previously and serve to demonstrate efficiency of the method. These solutions are of independent interest as they provide details in the exact solutions of the infinite-dimensional problems not seen before. In addition, they give new insight into the convergence of the computed collapse multipliers λ_h^*.

When comparing with other results we focus on the obtainable mesh size, which determines the discretization error, and on the quality of the computed collapse fields for stresses and flow. These fields are in general not uniquely determined. The computed fields may depend both on the finite element spaces and on the solution algorithm for the discrete problem. The degree of nonuniqueness depends on the finite element discretization, and, as mentioned in the introduction, interior-point methods typically give solution fields that are in better agreement with continuum mechanics than extreme-point methods.

In the computations reported here uniform grids are used. Adaptive mesh generation can and should be used, but our main goal is to demonstrate the strength of the discretization (step one) and the optimization algorithm (step two).

5.1. Plane strain example. The test problem in plane strain is described in [5, 12, 18] and [16, Example 11.1]. A rectangular block with thin symmetric cuts is being pulled by a uniform tensile force at the end faces. Figure 1 shows a cross section of the block and the reduction of the problem size by symmetry.

In [16, section 14] computed values for the collapse multiplier λ_h^* are reported for grids coarser than 15×15. Our results are in perfect agreement. We are not aware of any other computations with the divergence-free elements used here, but we can compare with results obtained with piecewise constant elements for the stresses and

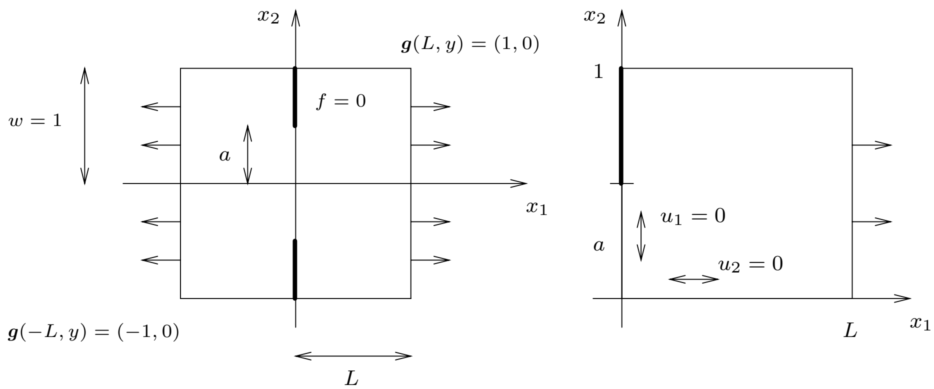

FIG. 1. *Geometry of the test problem in plane strain.*

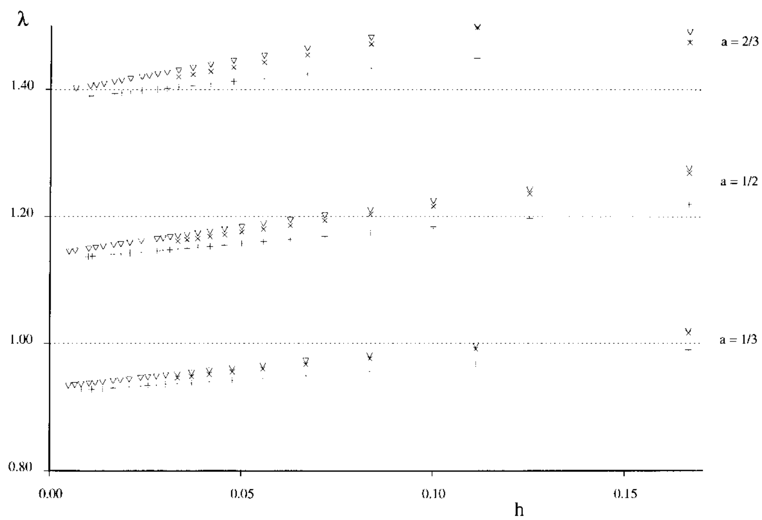

FIG. 2. *Computed values of the collapse multiplier for the test problem in Figure* 1 *for three values of the parameter a. $+$: Exact yield condition, divergence-free elements (new results). $\times$: Exact yield condition, constant-bilinear elements (from* [16, *section* 13]*). ∇ : Linearized yield condition, constant-bilinear elements (from* [5]*). For the mesh size h tending to zero the two sets of values for the exact yield condition but different finite elements should converge to the same limit. The values corresponding to the linearized yield condition is an upper bound, also in the limit.*

piecewise bilinear elements for the flow: in [5] the yield condition was linearized and the resulting LP was solved for grids up to 200×200, and in [16, section 13] the convex problem was solved on a 30×30 grid but with poorly determined stresses (the $\boldsymbol{x}$-variables). Figure 2 shows these results in comparison to the results obtained here for several values of the mesh size h and three values of the parameter a indicated in Figure 1. The values corresponding to the exact yield condition, but different finite element discretizations ($+$ and $\times$), converge to the same limit, as h tends to zero, while the values corresponding to the linearized yield condition are upper bounds, exceeding the values indicated by $\times$ by less than 2%.

The largest problem solved with our new method is for a 120×120 grid. However, since the elements of higher degree used here have four times as many degrees of

Table 1

Results and convergence analysis for the test problem in Figure 1, $L = 1$, $a = \frac{1}{3}$, $a = \frac{1}{2}$, and $a = \frac{2}{3}$. $k(h)$ is the computed convergence order, and $R(h)$ is the Richardson extrapolation to order 1.

	$a = 1/3$			$a = 1/2$			$a = 2/3$		
h^{-1}	$\lambda^*(h)$	$k(h)$	$R(h)$	$\lambda^*(h)$	$k(h)$	$R(h)$	$\lambda^*(h)$	$k(h)$	$R(h)$
6	0.9898			1.2193			1.4775		
12	0.9560		0.9221	1.1751		1.1308	1.4336		1.3896
18	0.9450	1.05	0.9230	1.1603	1.00	1.1309	1.4172	0.80	1.3844
24	0.9396	1.06	0.9235	1.1530	1.02	1.1311	1.4088	0.95	1.3838
30	0.9364	1.05	0.9237	1.1487	1.03	1.1313	1.4038	0.95	1.3835
36	0.9343	1.05	0.9238	1.1458	1.02	1.1314	1.4003	0.97	1.3833
42	0.9328	1.05	0.9239	1.1437	1.02	1.1314	1.3979	0.97	1.3832
48	0.9317	1.04	0.9239	1.1422	1.02	1.1314	1.3961	0.98	1.3832
54	0.9308	1.05	0.9240	1.1410	1.01	1.1315	1.3946	0.99	1.3832
60	0.9302	1.04	0.9240	1.1401	1.02	1.1315	1.3935	0.99	1.3831
90	0.9281	1.04	0.9241	1.1372	1.01	1.1315	1.3900	0.99	1.3831
99	0.9278	1.03	0.9241				1.3894	0.99	1.3831
100				1.1366	1.01	1.1315			
120	0.9271	1.03	0.9241						

freedom per node, a 120×120 grid corresponds in accuracy and problem size to a 240×240 grid for the two other methods in the comparison.

A selection of our results is shown in Table 1. We can solve the convex problem (2.11) at least as accurately as the LP solved in [5]. Table 1 shows, for $L = 1$ and the cases $a = \frac{1}{3}$, $a = \frac{1}{2}$, and $a = \frac{2}{3}$ (see Figure 1), some computed values for λ_h^*, the estimated convergence order based on these values (as described in [20]), and the estimate of λ^* obtained by Richardson extrapolation to order $k = 1$. These values are computed from the expression

$$R(h_i) = \frac{\alpha^k \lambda^*(h_i) - \lambda^*(h_{i-1})}{\alpha^k - 1}, \quad \alpha = \frac{h_{i-1}}{h_i},$$

where $h_i < h_{i-1}$ are two successive h-values in the table. The convergence analysis shows that the collapse multiplier λ_h^* converges in h with order 1, in the sense that the limit

$$\lim_{h \to 0} \frac{1}{h} (\lambda_h^* - \lambda^*) \tag{5.1}$$

exists. This was found not to be the case in [5], where the expression in (5.1) was bounded, but not convergent, as $h \to 0$. The explanation is that the finite element functions of higher degree used here approximate even the nonsmooth collapse fields in limit analysis better than piecewise linear or bilinear elements, although the convergence order is still $k = 1$. This makes it possible to estimate the discretization error and extrapolate to obtain an accuracy not seen before in limit analysis. For example, we claim that in the case $L = 1$ and $a = \frac{1}{3}$ we have $\lambda^* = 0.9241$ with all digits correct; i.e., the exact value of the infinite-dimensional problem (1.5) is determined to four decimal digits.

Figure 3 visualizes the computed collapse fields $\boldsymbol{u}_h^*$ and $\boldsymbol{\sigma}_h^*$ for the case $L = 1$, $a = \frac{1}{3}$, $h = \frac{1}{120}$. The vector field $\boldsymbol{u}_h^*$ in the solution is a displacement *rate* and could be plotted as a velocity field. However, it is standard to multiply the displacement rate by a suitable time scale and plot the resulting deformation. We have also chosen to do

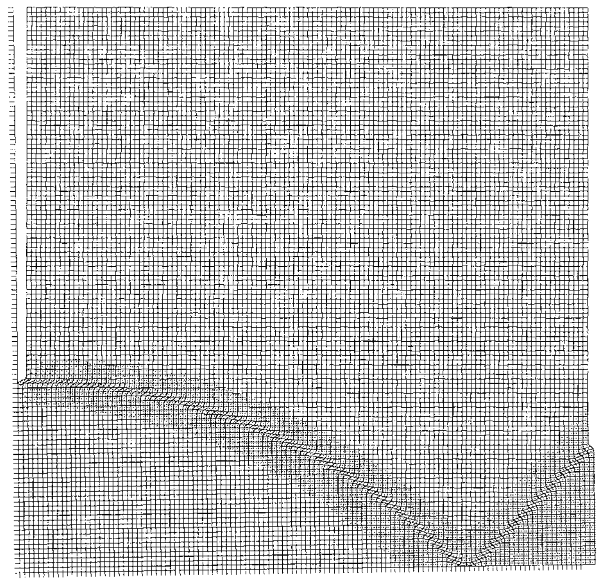

FIG. 3. *Collapse fields for the test problem in Figure* 1: $L = 1$, $a = \frac{1}{3}$, $h = \frac{1}{120}$.

this. It must be realized that deformations as large as those shown in the plots may never be observed and would in fact violate the model. The solution is a "snapshot" of the moment of collapse, and the plots only serve to visualize it. The deformed grid is not necessarily linear between nodes due to the higher order element functions. Rigid regions separated by a narrow slip zone are clearly recognizable. The nodes in the rigid region correspond to zero-norms in the sum (3.2). The nodes where the stress tensor satisfies the yield condition without slack are indicated by a small line segment indicating the direction of the vector (σ_1, σ_2) in (2.3). These nodes make up the plastified region. On this fine grid the direction can hardly be recognized, but the plastified region is clearly visible. In Figures 3 and 4, a node is considered plastified if the slack in the yield condition is less than 10^{-8}, but the picture is almost the same for all tolerances between 10^{-10} and 10^{-6}. The collapse fields are in agreement with those found in [5].

Figure 4 shows the collapse solution for the case $L = 2$, $a = \frac{1}{3}$, $h = \frac{1}{60}$. We see that the plastified region does not penetrate to the side $x = L$, suggesting that the collapse solution, in particular the limit multiplier λ^*, should remain the same if material is added to the rightmost rigid block. This was confirmed by actually computing the collapse solution for $L = 3$. For $h = \frac{1}{3}$ the values of λ_h^* for $L = 2$ and $L = 3$ are 1.258833 and 1.259579, respectively, but for $h \leq \frac{1}{12}$ the values of λ_h^* for $L = 2$ and $L = 3$ are identical. Through the same analysis as in Table 1 we find that for $L \geq 2$ and $a = \frac{1}{3}$ the limit multiplier is $\lambda^* = 1.130 \pm 0.001$.

The discrete problem corresponding to Figure 3 (the largest case solved) has $M = 58240$ variables (u-components) and $N = 116162$ dual variables (σ-components). There are 58081 terms in the sum (3.2), out of which more than 80% are zero in the optimal solution. The matrix $\boldsymbol{A}$ has $2,092,843$ nonzero elements. The computation used 79 hours of CPU-time on the CONVEX C3240 vector computer at Odense University. The accuracy measured in duality gap and lack of feasibility is about 10^{-8}. The CPU-time is considerable, but we have tried to test the limits of the method. In

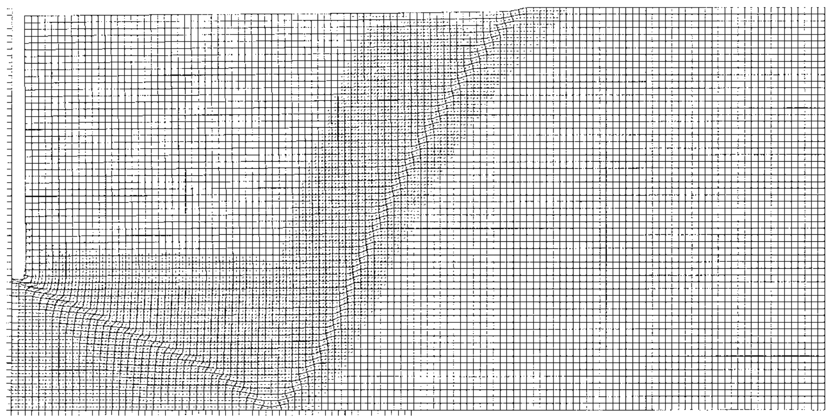

FIG. 4. *Collapse fields for the test problem in Figure* 1*:* $L = 2$*,* $a = \frac{1}{3}$*,* $h = \frac{1}{60}$*.*

our experience the bottleneck in limit analysis has not been the CPU-time or storage, but the deterioration of accuracy due to ill conditioning.

5.2. Plate bending example. The second application is the plate bending problem described in [19]: various combinations of simply supported/clamped and square/rectangular plates loaded by a uniform load or a point load at the center. In [19] results are reported for meshes up to 40×40 grids on the square plate (reduced to 20×20 by symmetry). We have results for 800×800 grids (reduced to 400×400 by symmetry). As in [19] piecewise bilinear finite element functions are used both for u and for the components of $\boldsymbol{m}$.

The largest case solved has $M = 160000$ variables, $N = 482403$ dual variables, and 160801 terms in the sum (4.10), and the matrix $\boldsymbol{A}$ has 3,390,400 nonzero entries. The computation used 5.7 hours of CPU-time (same computer as above). The duality gap was less than 10^{-8}, while infeasibilities (primal and dual) were less than 10^{-10}. This problem can be solved very efficiently because all terms in the sum of norms (4.10) are nonzero in the optimal solution. (This is in strong contrast to the first application in plane strain.) This was also essential for the results in [19] which were obtained using the smooth optimization algorithm by Goldfarb [24].

Our results agree with earlier results, but the fact that we are able to solve for much finer grids makes a better convergence analysis possible. This is done as described in [20]. Table 2 shows the value of λ_h^* for the simply supported square plate with uniform load. The computed convergence orders $k_1(h)$ in column 3 confirm that the error is of order 2. Column 4 shows the result $R_1(h)$ of Richardson extrapolation to order 2. We then estimate the order $k_2(h)$ of the error after extrapolation (column 5) and find the order 3. We expect the orders 2 and 3 for the two lowest order terms in the error for smooth solutions u^* and $\boldsymbol{m}^*$ with piecewise bilinear element functions. Finally, column 6 shows the result $R_2(h)$ of a second extrapolation, this time to order 3. We find the collapse multiplier for the continuous problem to be $\lambda^* = 25.019066$, correct to all digits.

For the uniformly loaded clamped plate the results are shown in Table 3. In the clamped case u^* has a singularity in the form of a so-called hinge along the boundary, resulting in a slower convergence. An analysis similar to the one for Table 2 indicates that the lowest two orders in the error are 1.5 and 2. For this case we find the value

Table 2
Results and convergence analysis for the simply supported, uniformly loaded square plate.

h^{-1}	$\lambda(h)$	$k_1(h)$	$R_1(h)$	$k_2(h)$	$R_2(h)$
12	24.86336954				
24	24.97645373		25.01414846		
36	24.99948059	1.84	25.01790208		25.01884048
48	25.00787151	1.89	25.01865984	2.56	25.01908081
60	25.01183455	1.92	25.01887995	3.06	25.01907256
72	25.01401352	1.94	25.01896572	3.09	25.01906865
84	25.01533804	1.95	25.01900594	3.06	25.01906738
96	25.01620269	1.96	25.01902721	3.07	25.01906672
108	25.01679808	1.97	25.01903955	3.03	25.01906653
120	25.01722540	1.97	25.01904713	3.09	25.01906623
200	25.01839885	1.98	25.01905892	2.99	25.01906628
300	25.01876857	1.98	25.01906435	3.03	25.01906621
400	25.01889849	1.99	25.01906553	3.03	25.01906619

Table 3
Results and convergence analysis for the clamped, uniformly loaded square plate.

h^{-1}	$\lambda(h)$	$k_1(h)$	$R_1(h)$	$k_2(h)$	$R_2(h)$
24	43.73575159				
36	43.91239918		44.12341810		
48	43.98660561	1.49	44.12412661		44.12484481
60	44.02590034	1.48	44.12474444	-0.52	44.12567757
72	44.04966063	1.48	44.12520184	0.33	44.12612043
96	44.07627631	1.48	44.12560108	1.38	44.12625923
120	44.09041730	1.47	44.12598832	0.31	44.12657317
200	44.10973000	1.46	44.12649948	0.49	44.12695585
400	44.12076200	1.47	44.12679560	2.20	44.12691930
800	44.12471300	1.48	44.12687387	2.28	44.12689996

$\lambda^* = 44.1269$ with uncertainty in the last digit only. This confirms the value found in [19, p. 180] but is in conflict with a conjecture in [1, p. 135], which implies a lower bound of 44.46. This discrepancy cannot possibly be explained by the discretization error. The computer programs used in the present work and in [19] were prepared completely independently, although both are based on the same conceptual method for discretizing the clamped plate by the finite element method. In [1, p. 135] the authors agree that "the computed results throw doubt on the validity of the conjecture." From our results we draw the conclusion that the conjecture is incorrect.

The collapse multiplier λ_h^* for a discrete point load is shown in Figure 5 for the following four cases: simply supported/clamped and square (1×1)/rectangular (1×2) plate. In the clamped case the values for the square and rectangular plate (white and black diamonds in Figure 5) overlap almost completely, and the difference tends to zero with h. In the plastic plate model a point load is only admissible as a limit of concentrated loads and may be approximated by a sequence of discrete loads "shrinking" with h as well as by discrete point loads (for details see [19, p. 181]). Since the error in the simply supported case is quite large (white and black circles in Figure 5), we have also approximated the point load by a sequence of unit loads distributed uniformly on a central square of side h (white squares in Figure 5). As mentioned in [19] this approximation must yield the same limit as the discrete point loads if the concept of a point load is valid. This appears to be the case.

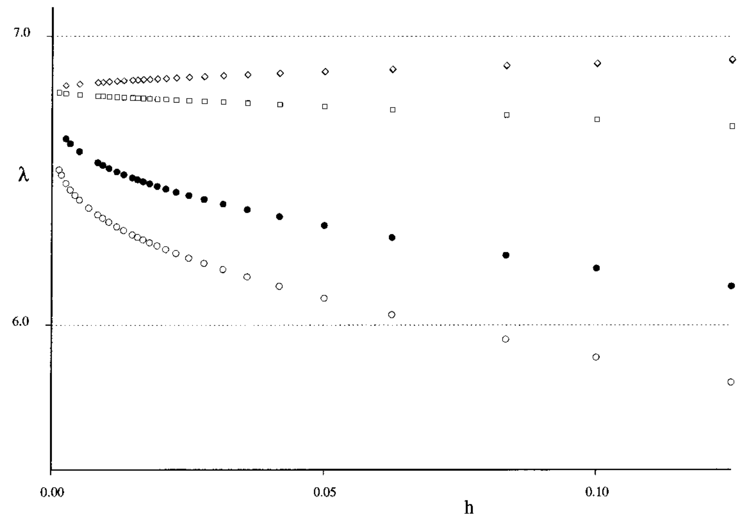

FIG. 5. *Computed values of the collapse multiplier for various approximations of a point load:* ○ : *Simply supported square plate; discrete point load.* ● : *Simply supported rectangular plate; discrete point load.* ◊ : *Clamped square plate; discrete point load.* ⧫ : *Clamped rectangular plate; discrete point load (almost hidden by* ◊*).* □ : *Simply supported square plate; load concentrated uniformly at a central square of side h.*

Our results confirm (although not beyond any doubt) a claim made in [19]: the five sequences of discrete values λ_h^* in Figure 5 converge to the same limit. This means that the limit multiplier for a point load does not depend on shape or support of the plate. We find this value to be $\lambda^* = 6.82 \pm 0.01$.

Convergence analysis of the results shows that, for all the above-mentioned discretizations of a point load, λ_h^* converges more slowly than h^k for any power $k > 0$, as $h \to 0$. The collapse solutions in Figure 6 (simply supported plate) and Figure 7 (clamped plate) clearly indicate why: the deformation is singular. On the other hand, our results indicate that the deformation converges as $h \to 0$, which means that the peak at the center is finite. Solutions for coarser grids could not show this. The solutions shown in Figures 6 and 7 are for a 200×200 grid (by symmetry the computation is reduced to a 100×100 grid). The largest case solved is for an 800×800 grid, but this is too fine to plot.

6. Conclusion. We have developed an efficient method for the computation of collapse states in limit analysis with the von Mises yield criterion. The method can be used with standard finite element functions in applications with bounded yield set, such as plate bending and plane stress. In applications with unbounded yield set, such as plane strain or the general problem in three space dimensions where the plastic flow is incompressible, we use finite element functions that are differentiable across element boundaries.

An alternative approach, avoiding the use of differentiable element functions, is to impose the condition of incompressibility as a large set of constraints in the minimum-sum-of-norms problem. The resulting constrained problem must then be solved with an efficiency that is comparable to the efficiency obtained here. In future work we shall pursue this direction.

Acknowledgment. The authors are grateful to Ken Kortanek for suggesting this collaboration.

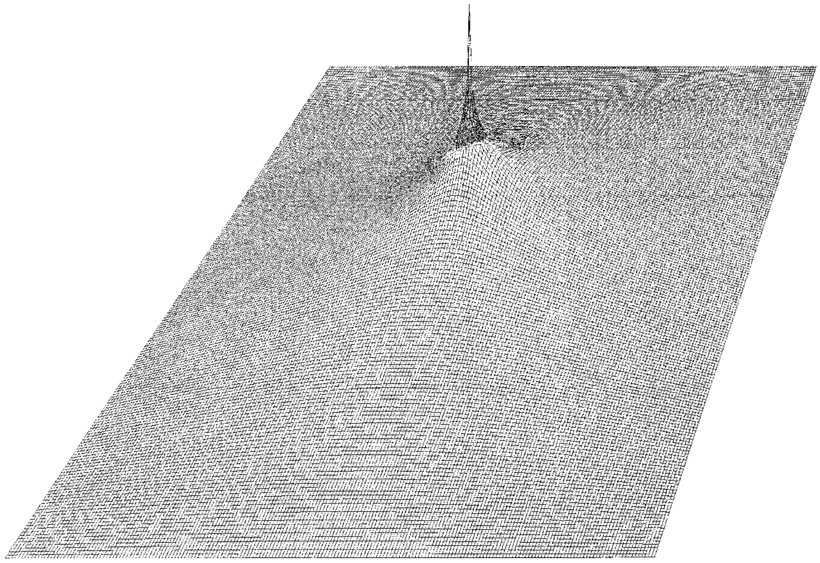

Fig. 6. *Simply supported square plate with point load.*

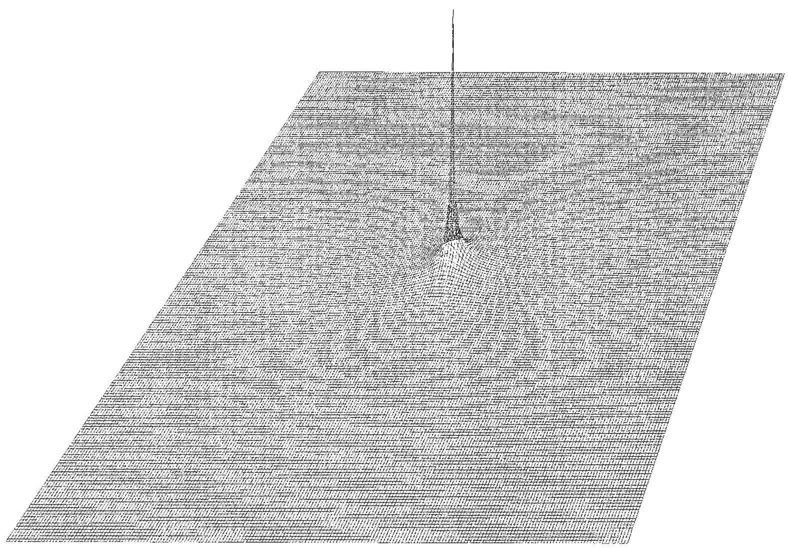

Fig. 7. *Clamped square plate with point load.*

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