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U.S.S.R. COMPUTATIONAL MATHEMATICS AND MATHEMATICAL PHYSICS

VOLUME 6, 1966

PERGAMON PRESS

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U.S.S.R. COMPUTATIONAL MATHEMATICS AND MATHEMATICAL PHYSICS

Cover-to-Cover translation of
Zhurnal vychislitel'noi matematiki i matematicheskoi fiziki
Published bi-monthly by Pergamon Press

Annual subscription rate $140
Subscriptions are only serviced after payment has been received and are not accepted for less than one year.
Orders should be sent to the distributors at either of the following addresses:
Headington Hill Hall, Oxford, OX3 0BW
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PUBLISHED BY
PERGAMON PRESS LTD
OXFORD · LONDON · EDINBURGH · PARIS · NEW YORK

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- 178, 1961.

PERTURBATION THEORY OF NON-CONJUGATE OPERATORS*

V.B. LIDSKII

Moscow

(*Received* 9 *April* 1965)

LET $\mathbf{A}$ and $\mathbf{B}$ be two linear operators in N-dimensional unitary space $\mathfrak{H}$. Let λ_0 be the eigenvalue of $\mathbf{A}$ to which corresponds the radical subspace S_{λ_0}.

Let us consider the operator

$$\mathbf{C}(\varepsilon) = \mathbf{A} + \varepsilon\mathbf{B},$$

where ε is a small parameter. The question of the behaviour of the eigenvalues $\lambda(\varepsilon)$ of the operator $\mathbf{C}(\varepsilon)$, near λ_0, with Jordan cells to the operator $\mathbf{A}$, has been considered in a number of papers [1 - 3]. In particular, in [2] a general theorem is established on the expansion of the eigenvalues and eigenvectors of $\mathbf{C}(\varepsilon)$ in a series in fractional powers of ε.

In the present paper we apply to the given problem a method, which is used in [1], and not only obtain a simple proof of the theorem on the expansion in a series in powers of $\varepsilon^{1/l}$, where l is the order of the Jordan cell, but also write out explicitly the secular equations for first approximations to the eigenvalues (see (1.9)), and a system for the determination of the zero approximations to the eigenvectors of the perturbed operator (see (1.12)). We shall not consider further approximations in the present paper.

All the results can be transferred without change to the case of the operator

$$\mathbf{C}(\varepsilon) = \mathbf{A} + \varepsilon\mathbf{B} + \sum_{k=2}^{\infty} \varepsilon^k \mathbf{B}_k,$$

* *Zh. vȳchisl. Mat. mat. Fiz.* 6, 1, 52 - 60, 1965.

incorrect.
Should be 1966.

where the series converges if $|\varepsilon| \leqslant \varepsilon_0$.

The basis of our reasoning is the following simple idea. The secular equation

$$P(\lambda, \varepsilon) = \mathrm{Det}\,(\lambda \mathbf{E}_N - \mathbf{C}(\varepsilon)) = 0 \qquad (0.1)$$

with Jordan calls of order l to the operator $\mathbf{A}$ can, by the substitution

$$\frac{\lambda - \lambda_0}{z} = \mu, \qquad z = \varepsilon^{1/l} \qquad (0.2)$$

be reduced to the form

$$\tilde{P}(\mu, z) = 0, \qquad (0.3)$$

where the left-hand side is an integral polynomial in powers of μ and z. If the equation $\tilde{P}(\mu, 0) = 0$ has a simple root, then by the well-known theorem of the theory of functions [4, pp. 666 - 670] equation (0.3) determines an analytic function $\mu(z)$. Hence the expansion of $\lambda(\varepsilon)$ in powers of $\varepsilon^{1/l}$ also follows. The expansion of the eigenvectors in a series follows from the fact that, as the components of the eigenvector, we can choose the minors of the matrix $\lambda(\varepsilon)E_N - C(\varepsilon)$.

1. We now make some preliminary remarks. Let A be the matrix of the operator $\mathbf{A}$ in some fixed orthonormal basis, and let

$$e_1, e_2, \ldots, e_N \qquad (1.1)$$

be the Jordan basis of $\mathbf{A}$. Then $A = KJK^{-1}$, where J is a Jordan matrix, and in the columns of K are the coordinates of the vectors (1.1) in the original orthonormal basis. We have $A^* = K^{-1*}J^*K^*$.

We shall denote by

$$\omega_1, \omega_2, \ldots, \omega_N \qquad (1.2)$$

the vectors whose coordinates are situated in the columns of the matrix K^{-1*}. By virtue of the obvious equality $(K^{-1*})^*K = E_N$ the vectors (1.1) - (1.2) form a biorthogonal system

$$(\omega_\alpha, e_\beta) = \delta_{\alpha\beta}.$$

We now note that if $e_1, e_2, \ldots, e_m$, is a Jordan chain, belonging to the basis (1.1), i.e. $\mathbf{A}e_1 = \lambda_0 e_1$, $\mathbf{A}e_2 = \lambda_0 e_2 + e_1$, $\ldots$, $\mathbf{A}e_{m_1} = \lambda_0 e_{m_1} + e_{m_1-1}$, the vectors $\omega_1, \omega_2, \ldots, \omega_{m_1}$ form the Jordan chain of the operator $\mathbf{A}^*$:

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$$\mathbf{A}^*\omega_{m_1} = \bar{\lambda}_0\omega_{m_1},\ \mathbf{A}^*\omega_{m_1-1} = \bar{\lambda}_0\omega_{m_1-1} + \omega_{m_1}, \ldots, \mathbf{A}^*\omega_2 = \bar{\lambda}\omega_2 + \omega_1$$

(the eigenvector is ω_{m_1}); the last equalities follow directly from the formula $A^*K^{*-1} = K^{*-1}J^*$ by virtue of the special form of the matrix J^*. Such a correspondence also holds, obviously, for other chains of the bases (1.1) and (1.2). Thus (1.2) is a Jordan basis of $\mathbf{A}^*$.

Suppose that the radical subspace S_{λ_0} of the operator $\mathbf{A}$ splits into q_p cyclic subspaces of dimensionalities

$$l_1 = m_1 = m_2 = \ldots = m_{q_1} > \underbrace{m_{q_1+1} = \ldots = m_{q_2}}_{l_2} > \ldots > \underbrace{m_{q_{p-1}+1} = \ldots = m_{q_p}}_{l_p}.$$

The eigenvectors of the operator $\mathbf{A}$, belonging to S_{λ_0} and occurring in the basis (1.1), will be denoted by

$$f_1, f_2, \ldots, f_{q_p}.$$

The numeration of the eigenvectors is in accordance with that of the corresponding cyclic subspaces.

The eigenvectors of the operator $\mathbf{A}^*$ of the basis (1.2), which belong to the corresponding radical subspace $S_{\bar{\lambda}_0}$, will be denoted by

$$g_1, g_2, \ldots, g_{q_p}.$$

Note that with the given numeration

$$f_j = e_{\beta_{j-1}}, \qquad (1.3)$$

where

$$\beta_{j-1} = m_1 + m_2 + \ldots + m_{j-1} + 1, \qquad m_0 = 0, \qquad (1.3')$$

and

$$g_i = \omega_{\alpha_i}, \qquad (1.4)$$

where

$$\alpha_i = m_1 + m_2 + \ldots + m_i. \qquad (1.4')$$

For future work it is convenient to denote the dimensionality of the cyclic subspaces of dimensionality equal to that of the first group by l_1, of the second by l_2 etc., and the multiplicities of the cyclic

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subspaces of equal dimensionality by $r_1, r_2, \dots, r_p$ respectively.

Thus

$$r_s = q_s - q_{s-1} \qquad (s = 1, 2, \dots, p), \qquad q_0 = 0. \tag{1.5}$$

We now introduce into the discussion the following square matrices*:

$$B_{q_s, q_s} = \|b_{ij}\|, \qquad i, j = 1, \dots, q_s, \qquad s = 1, 2, \dots, p,$$

where

$$b_{ij} = (\mathbf{B} f_j, g_i).$$

We denote by E_{q_s, r_s} the square matrix of order q_s

$$E_{q_s, r_s} = \left\| \begin{matrix} 0 & 0 \\ 0 & E_{r_s} \end{matrix} \right\|, \qquad \left.\begin{matrix} \}\, q_{s-1} \\ \}\, r_s \end{matrix}\right\}\, q_s$$

where E_{r_s} is a unit matrix of order r_s.

We now formulate the basic results of this paper.

Theorem 1

Suppose that for some s**

$$\operatorname{Det} B_{q_{s-1}, q_{s-1}} \neq 0, \qquad \operatorname{Det} B_{q_s q_s} \neq 0 \tag{1.6}$$

$(1 \leqslant s \leqslant p)$. Then the perturbed operator $\mathbf{C}(\varepsilon) = \mathbf{A} + \varepsilon\mathbf{B}$ possesses eigenvalues

$$\lambda_t^{(s)}(\varepsilon) = \lambda_0 + \varepsilon^{1/l_s} \sqrt[l_s]{\sigma_t^{(s)}} + o(\varepsilon^{1/l_s}), \qquad t = 1, 2, \dots, r_s, \tag{1.7}$$

where the numbers

$$\sigma_1^{(s)}, \sigma_2^{(s)}, \dots, \sigma_{r_s}^{(s)} \tag{1.8}$$

are distinct from zero and are found from the equation

$$\operatorname{Det}(B_{q_s, q_s} - \sigma E_{q_s, r_s}) = 0; \tag{1.9}$$

---

* p is the number of groups of cyclic subspaces of the same dimensionality.

** s is the number of the group of cyclic subspaces of equal dimensionality; r_s the number of cyclic subspaces belonging to a given group.

and in the formula (1.7) all the complex values of ε^{1/l_s} are taken.

If, in addition, all the roots of (1.8) are different, then the eigenvalues of (1.7) can be expanded in power series of the form

$$\lambda_t^{(s)}(\varepsilon) = \lambda_0 + \varepsilon^{1/l_s} \sqrt[l_s]{\sigma_t^{(s)}} + \sum_{k=2}^{\infty} \varepsilon^{k/l_s} a_k^{(s,t)}, \qquad t = 1, 2, \dots, r_s. \tag{1.10}$$

Theorem 2

Let

$$\operatorname{Det} B_{q_k, q_k} \neq 0, \qquad k = 1, 2, \dots, p;$$

and for some s let the roots (1.8) of the secular equation (1.9) be different; then for sufficiently small $\varepsilon \neq 0$ the eigenvalues (1.10) of the operator $\mathbf{C}(\varepsilon)$ are simple and the eigenvectors corresponding to them can be put in a series

$$\varphi_t^{(s)}(\varepsilon) = \varphi_0^{(s,t)} + \sum_{k=1}^{\infty} \varphi_k^{(s,t)} \varepsilon^{k/l_s},$$

where

$$\varphi_0^{(s,t)} = \sum_{j=1}^{q_s} c_j^{(s,t)} f_j \neq 0, \tag{1.11}$$

and the coordinates $c_j^{(s,t)}$ $(j = 1, 2, \dots, q_s)$ are found from the system of equations*

$$(B_{q_s, q_s} - \sigma_t^{(s)} E_{q_s, r_s})c = 0, \tag{1.12}$$

where c is a vector-column.

2. *Proof of Theorem 1.*

1. We first reduce equation (0.1) to the form (0.3). To simplify the

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* In view of the simplicity of the root $\sigma_t^{(s)}$, the system (1.12) possesses one solution within the accuracy of the multiplies.

notation we shall assume that λ_0 is a unique eigenvalue of the operator $\mathbf{A}$ and, consequently, S_{λ_0} coincides with the whole space $\mathfrak{H}$. This assumption, as will be seen from what follows, does not influence the results obtained.

The matrix J now assumes the form

$$J = \begin{Vmatrix} R_{m_1} & & & 0 \\ & R_{m_2} & & \\ & & \ddots & \\ 0 & & & R_{m_{q_p}} \end{Vmatrix},$$

where R_{m_i} are Jordan cells. We shall denote the matrices of the operators $C(\varepsilon)$ and B in the basis (1.1) by $\mathbf{C}(\varepsilon)$ and $\mathbf{B}$. In particular

$$B = \| (\mathbf{B}e_\beta, \omega_\alpha) \|, \qquad \alpha, \beta = 1, \ldots, N. \tag{2.1}$$

We shall assume that $C(\lambda, \varepsilon) = \lambda E_N - C(\varepsilon)$, and consider the equation

$$P(\lambda, \varepsilon) = \operatorname{Det} C(\lambda, \varepsilon) = \operatorname{Det} (\lambda E_N - J - \varepsilon B).$$

We transform the matrix $C(\lambda, \varepsilon)$ by introducing new variables μ and by formulae (0.2) with $l = l_s$. This transformation is of an elementary character (the division of rows and columns of the matrix $C(\lambda, \varepsilon)$ by suitable powers of z). To shorten the calculations we introduce the following diagonal matrices

$$D_{m_i} = \operatorname{diag}(1, z, \ldots, z^{m_i-1}),$$
$$F_{m_i} = \operatorname{diag}(z^{-1}, z^{-2}, \ldots, z^{-m_i})$$

for* $m_i \leqslant l_s$ and

$$D_{m_i} = \operatorname{diag}(\underbrace{1, 1, \ldots, 1}_{m_i - l_s}, 1, z, \ldots, z^{l_s-1}), \qquad z^{(l_s-1)}$$
$$F_{m_i} = \operatorname{diag}(\underbrace{1, 1, \ldots, 1}_{m_i - l_s}, z^{-1}, z^{-2}, \ldots, z^{-l_s})$$

for $m_i > l_s$.

After this we introduce diagonal matrices $D(z)$ and $F(z)$ of order N,

* Here and everywhere below s and l_s are fixed in accordance with the condition of Theorem 1.

which have the same block structure as the matrix J, with the only difference that the blocks R_{m_i} are replaced by D_{m_i} and F_{m_i} respectively. We shall also consider the matrix

$$H(\mu, z) = F(z) C(\lambda, \varepsilon) D(z) = H^{(1)}(\mu, z) + H^{(2)}(z), \tag{2.2}$$

where

$$H^{(1)}(\mu, z) = F(z)(\lambda E_N - J) D(z),$$
$$H^{(2)}(z) = -z^{l_s} F(z) B D(z).$$

We shall prove that the left-hand side of the equation

$$\mathcal{P}(\mu, z) = \operatorname{Det} H(\mu, z) = 0 \tag{2.3}$$

is a polynomial in μ and z. Let us assume that

$$I_{m_i} = \begin{Vmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{Vmatrix}.$$

Then $\lambda E_{m_i} - R_{m_i} = (\lambda - \lambda_0) E_{m_i} - I_{m_i}$.

It is easy to verify that for all m_i

$$F_{m_i}(z) I_{m_i} D_{m_i}(z) = I_{m_i},$$

and for $m_i \leqslant l_s$

$$F_{m_i}(z) D_{m_i}(z) = z^{-1} E_{m_i}.$$

Therefore for all $m_i \leqslant l_s$ we have for the diagonal blocks of $H^{(1)}(\mu, z)$

$$H^{(1)}_{m_i}(\mu, z) = F_{m_i}(z)[(\lambda - \lambda_0) E_{m_i} - I_{m_i}] D_{m_i}(z) = \mu E_{m_i} - I_{m_i}, \tag{2.4}$$

and for $m_i > l_s$

$$H^{(1)}_{m_i}(\mu, z) = \operatorname{diag}(\underbrace{z\mu, z\mu, \ldots, z\mu}_{m_i - l_s}, \mu, \mu, \ldots, \mu) - I_{m_i}. \tag{2.4'}$$

We now turn to the matrix $H^{(2)}(z)$. We divide the matrix B (see (2.1)) into blocks of dimensions $m_i \times m_j$

$$B = \| B_{m_i m_j} \|, \qquad i, j = 1, \ldots, q_p,$$

in accordance with the block structure of J, D and F.

On carrying out multiplication by blocks we obtain

$$H^{(2)}_{m_i, m_j}(z) = -z^{l_s} F_{m_i}(z) B_{m_i, m_j} D_{m_j}(z). \tag{2.5}$$

Since the matrices $z^{l_s} F_{m_i}(z)$ and $D_{m_j}(z)$ contain only non-negative powers of z, it follows from formulae (2.2) - (2.5) that $P(\mu, z)$ is a polynomial in μ and z.

2. We now find $\tilde{P}(\mu, 0)$ and show that the equation

$$\tilde{P}(\mu, 0) = \det \tilde{H}(\mu, 0) = 0 \tag{2.6}$$

can be reduced to the form (1.9) on the basis of Laplace's theorem. For this we turn to the matrix

$$H(\mu, 0) = H^{(1)}(\mu, 0) + H^{(2)}(0) = \| H_{m_i m_j}(\mu, 0) \|, \quad i, j = 1, \ldots, q_p.$$

$H^{(1)}(\mu, 0)$ has a block-diagonal structure and its diagonal blocks are defined by formulae (2.4) and (2.4').

From formula (2.5), taking into account the form of the diagonal matrices*

$$z^{l_s} F_{m_i}(z)\,|_{z=0} \quad \text{и} \quad D_{m_j}(z)\,|_{z=0},$$

we easily discover that

$$H^{(2)}_{m_i, m_j}(\mu, 0) = \left\| \begin{matrix} 0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\ -b_{ij} & * & \underbrace{* & \ldots & *}_{m_j - l_s} & \ldots & 0 \end{matrix} \right\|, \quad m_i \geqslant l_s, \quad m_j \geqslant l_s; \tag{2.7}$$

$$H^{(2)}_{m_i, m_j}(\mu, 0) = 0, \quad m_i < l_s.$$

In formula (2.7), according to (1.3) and (1.4)

$$b_{ij} = (\mathbf{B} f_j, g_i),$$

and the elements marked by asterisks are not essential for us. Formula (2.7) is retained also for $m_i \geqslant l_s$, $m_j < l_s$, and here the asterisks are

---

* If $m_i < l_s$ the matrix $z^{l_s} F_{m_i}(z)|_{z=0}$ is zero, and if $m_i \geqslant l_s$ it has only one element different from zero.

replaced with zeros.

We make one more observation which we use to simplify the determinant (2.6). Let $H_{m_i m_i}(\mu, 0)$ be a diagonal block of $H(\mu, 0)$ of order $m_i = l_s$. According to (2.4) and (2.7) it has the form

$$H_{m_i, m_i}(\mu, 0) = \left\| \begin{matrix} \mu & -1 & 0 & \ldots & 0 \\ 0 & \mu & -1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & -1 \\ -b_{ii} & 0 & 0 & \ldots & \mu \end{matrix} \right\|, \tag{2.8}$$

$$q_{s-1} + 1 \leqslant i \leqslant q_s.$$

We multiply the second column of this minor by μ, the third by μ^2 etc., and the last by $\mu^{l_s - 1}$ and add them to the first. As a result block (2.8) is transformed into

$$\tilde{H}_{m_i, m_i}(\mu, 0) = \left\| \begin{matrix} 0 & -1 & 0 & \ldots & 0 \\ 0 & \mu & -1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & -1 \\ -b_{ii} + \mu^{l_s} & 0 & 0 & \ldots & \mu \end{matrix} \right\|. \tag{2.9}$$

Since all the remaining elements of the matrix $H(\mu, 0)$, situated in the columns, which were subjected to multiplication by powers of μ, are equal to zero by virtue of (2.7), and because $m_i = m_j = l_s$, the determinant (2.6) is unchanged if the blocks (2.8) are replaced by the blocks (2.9). We shall denote the matrix which is thus produced by $\tilde{H}(\mu, 0)$.

It is not difficult to see that in the determinant Det $\tilde{H}(\mu, 0)$ we can delete all the rows with numbers

$$\alpha_{i-1} + 1 \leqslant \alpha \leqslant \alpha_i - 1, \quad 1 \leqslant i \leqslant q_s, \tag{2.10}$$

$$\alpha_{q_s} + 1 \leqslant \alpha \leqslant N \tag{2.10'}$$

and columns with numbers

$$\beta_{j-1} + 1 \leqslant \beta \leqslant \beta_j, \quad i \leqslant j \leqslant q_s, \tag{2.11}$$

$$\beta_{q_s} + 1 \leqslant \beta \leqslant N, \tag{2.11'}$$

where α_i and β_j are defined by the formulae (1.4') and (1.3'). The fact of the matter is that in the rows of (2.10) there is only one minor of maximum order in each which is not zero. These minors are situated in the columns of (2.11) respectively. In the first case the minors are equal to ± 1, and in the second to μ^γ, where $\gamma = N - \alpha_{q_s}$. Therefore equation (2.3) assumes the form (see (1.9))

$$\mu^\gamma \operatorname{Det}(B_{q_s, q_s} - \sigma E_{q_s, r_s}) = 0, \tag{2.12}$$

where $\sigma = \mu^{l_s}$.

By virtue of condition (1.6) this equation is of degree r_s in σ and has r_s roots

$$\sigma_1^{(s)}, \sigma_2^{(s)}, \ldots, \sigma_{r_s}^{(s)}. \tag{2.13}$$

which are different from zero. From the theory of functions it is known that equation (2.3) has as roots

$$\mu_t^{(s)}(z) = \sqrt[l_s]{\sigma_t^{(s)}} + o(1), \qquad t = 1, 2, \ldots, r_s. \tag{2.14}$$

Hence by virtue of (0.2) with $l = l_s$ (1.7) also follows.

If the roots (2.13) are different the functions $\mu_t^{(s)}(z)$ can be expanded in power series in integral powers of z.

Hence follows the existence of r_s convergent expansions (1.10). This proves Theorem 1 completely.

3. Turning to the proof of Theorem 2, we note that if the conditions of this theorem are satisfied the eigenvalues of the operator, determined by the series (1.10), are simple for small $\varepsilon \neq 0$ (in view of (1.7) they cannot coincide with the eigenvalues obtained as a result of the splitting of cells of orders $l_{s'} \neq l_s$).

If we take a series of N simultaneously non-vanishing cofactors of the elements of the row of the matrix $\lambda_t^{(s)}(\varepsilon)E_N - C(\varepsilon)$, then, reducing it, perhaps, by some power ε^{1/l_s}, we obtain a convergent expansion for the eigenvector*

$$\varphi_t^{(s)}(\varepsilon) = \varphi_0^{(s,t)} + \sum_{k=1}^{\infty} \varphi_k^{(s,t)} \varepsilon^{k/l_s}, \qquad \varphi_0^{(s,t)} \neq 0.$$

In conclusion we find $\varphi_0^{(s,t)}$. Let

* Below we use the same symbol to denote the vector and the column of its coordinates in the basis (1.1).

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$$h^{(s,t)}(z) = h_0^{(s,t)} + \sum_{k=1} h_k^{(s,t)} z^k, \qquad h_0^{(s,t)} \neq 0, \tag{3.1}$$

be the vector which is annihilated by the matrix* $H(z) \equiv H(\mu_t^{(s)}(z), z)$.

We have

$$H(z)h(z) = F(z)C(\lambda_t^{(s)}(\varepsilon), \varepsilon)D(z)h(z) = 0. \tag{3.2}$$

Since the matrix $F(z)$ $(z \neq 0)$ is non-degenerate, the vector $D(z)h(z)$ is annihilated by the matrix $C(\lambda_t^{(s)}(\varepsilon), \varepsilon)$ and, consequently, is an eigenvector of $C(\varepsilon)$.

To find the vector $D(0)h(0)$ we must find the non-zero solution of the system of equations

$$H(0)h = 0$$

and apply to it the matrix $D(0)$. This system is easily simplified by the same method by which the determinant (2.6) was transformed, and is reduced to system (1.12). Using also the simple form $D(0)$, it is not difficult to arrive at the conclusion that the vector $D(0)h(0) \neq 0$ and has the form

$$\sum_{j=1}^{q_s} c_j^{(s,t)} f_j,$$

where $c_j^{(s,t)}$ $(j = 1, 2, \ldots, q_s)$ satisfies system (1.12). Assuming that $\varphi_0^{(s,t)} = D(0)h(0)$ we arrive at the formula (1.11). Theorem 2 has now been proved.

4. We now make a few more observations.

1. If we had not assumed earlier on that the radical subspace S_{λ_0} coincide with the whole space $\mathfrak{H}$, then, as is usual in perturbation theory, we must introduce the projection operator, analytically, depending on ε,

$$\mathbf{P}(\varepsilon) = \frac{1}{2\pi i} \int_{C_\delta} (z\mathbf{E}_N - \mathbf{A} - \varepsilon\mathbf{B})^{-1} dz = \mathbf{P}(0) + \sum_{k=1}^{\infty} \varepsilon^k \mathbf{P}_k,$$

* In view of (2.2) the ranks of the matrices $H(z)$ and $C(\lambda_t^{(s)}(\varepsilon), \varepsilon)$ coincide if $z \neq 0$, and consequently are $N - 1$. Therefore the vector (3.1) is unique to within a scalar multiplier.

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where C_δ is a circle of sufficiently small radius with centre at the point λ_0. Here

$$\mathbf{P}(0) = \frac{1}{2\pi i} \oint_{C_\delta} (z\mathbf{E}_N - \mathbf{A})^{-1}\, dz$$

is the operator for projection onto the subspace S_{λ_0}. It is easy to prove that the operator

$$\mathbf{Q}(\varepsilon) = \mathbf{P}(\varepsilon)\mathbf{P}(0) = \mathbf{P}(\varepsilon)[\mathbf{E}_N - (\mathbf{P}(\varepsilon) - \mathbf{P}(0))]\mathbf{P}(0)$$

maps S_{λ_0} in a one-to-one manner onto the invariant subspace $S(\varepsilon) = \mathbf{P}(\varepsilon)\mathfrak{H}$ of the operator $\mathbf{C}(\varepsilon)$. Therefore the operator $\mathbf{Q}(\varepsilon)$ has an inverse $\mathbf{Q}^{-1}(\varepsilon)$, which can be put in the form $\mathbf{P}(0)[\mathbf{E}_N - (\mathbf{P}(\varepsilon) - \mathbf{P}(0))]^{-1}\mathbf{P}(\varepsilon)$ and depends analytically on ε. After this the problem is reduced to that already considered for the operator $\tilde{\mathbf{C}}(\varepsilon) = \mathbf{Q}^{-1}(\varepsilon)\mathbf{C}(\varepsilon)\mathbf{Q}(\varepsilon)$, which acts in S_{λ_0}. The fact is that

$$\tilde{\mathbf{C}}(\varepsilon) = \mathbf{P}(0)\mathbf{A}\mathbf{P}(0) + \varepsilon\mathbf{P}(0)\mathbf{B}\mathbf{P}(0) + \sum_{k=2}^{\infty} \varepsilon^k \mathbf{P}(0)\mathbf{B}_k\mathbf{P}(0),$$

and the unperturbed operator $\mathbf{P}(0)\mathbf{A}\mathbf{P}(0)$ has a unique eigenvalue λ_0. All the formulae in Theorems 1 and 2 are preserved here.

The given method enables us without alteration, to extend the results of Theorems 1 and 2 to operators, which act in Hilbert space, in the case where the operator $\mathbf{A}$ possesses a finite-dimensional, normally detached, radical subspace [5].

2. Conditions (1.6) are clearly essential for Theorem 1 to be valid. For instance, for an operator with matrix

$$C(\varepsilon) = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} + \varepsilon \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{Vmatrix}$$

the characteristic polynomial has the form $\lambda^3 - \varepsilon^2$, and, consequently, the eigenvalues are

$$\lambda_k = |\varepsilon|^{2/3} e^{i2k\pi/3} \qquad (k = 0, 1, 2),$$

when the dimensionalities of the cyclic subspaces of the unperturbed operator are 2 and 1.

3. We now give a practical method of constructing the matrix $B_{q_s q_s}$ (see (1.9)), if the matrices of the operators $\mathbf{A}$ and $\mathbf{B}$ in the Jordan basis (1.1) are known. We must introduce the matrix

$$\lambda E_N - J$$

and leave the rows and columns of B, at whose intersection are situated the blocks of the matrix (4.1) of orders $\leqslant l_s$, which correspond to λ_0.

Then we must consider all the rows and all the columns of the matrix (4.1), which each contain two elements different from zero, and delete the corresponding rows and columns of B. The remaining matrix is $B_{q_s q_s}$.

4. The identity (3.2) can be used to find the following terms of the expansions of the functions $\mu_t^{(s)}(z)$ and $h^{(s,t)}(z)$ in powers of z, since $\mu_t^{(s)}(x)$ is a simple root of equation (2.2).

Translated by H.F. Cleaves

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Su-Huan Chen, Zhong-Sheng Liu et al.

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vol. 9, p. 427-438, 1993