Domain Adaptation
Theory and Algorithms

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Standard Learning Assumptions

- IID assumption.
- Same distribution for training and test.
- Distributions fixed over time.
Modern Large-Scale Data Sets

Real-world applications:

- Sample points are not drawn IID.
- Training sample is biased.
- Training points with uncertain labels.
- Multiple training sources.
- Distribution may drift with time.

These problems must be addressed for learning to be effective.
Domain Adaptation - Problem

- **Input:**
  - Labeled data from source domain.
  - Unlabeled data from target domain.

- **Problem:** use labeled and unlabeled data to derive hypothesis $h$ with good performance on target domain.
  - Thus, harder generalization problem than standard learning problem!
Domain Adaptation - Examples

- Sentiment analysis:
  - appraisal information for some domains, e.g., movies, books, music, restaurants.
  - but no labeled information for travel.

- Language modeling, part-of-speech tagging, parsing.

- Speech recognition.

- Computer vision.
Related Work

- **Single-source adaptation:**
  - language modeling, probabilistic parsers, maxent models: source domain used to define a prior.
  - relation between adaptation and the $d_A$ distance [Ben-David et al. (2006) and Blitzer et al. (2007)].

- **Multiple-source:**
  - same input distribution, but different labels [Crammer et al. (2005, 2006)].
  - theoretical analysis and method for multiple-source adaptation [Mansour et al. (2008)].
Existing Techniques?

- Importance weighting technique.
  - applications, e.g., sample bias correction.
  - experiments: method does not always work.
  - can lead to performance degradation!
  - recent new analysis and learning guarantees (Cortes, Mansour, and MM, NIPS 2010).
This Talk

- Domain adaptation problem
- Discrepancy distance
- Theoretical guarantees
- Algorithm
- Experiments
Learning Set-up

- **Distributions**: source $Q$, target $P$.
- **Target function(s)**: $f$, or $f_Q$ and $f_P$.
- **Input**: labeled sample drawn from $Q$, unlabeled sample drawn from $P$.
- **Problem**: find hypothesis $h$ with small expected loss with respect to distribution $P$,

$$
\mathcal{L}_P(h, f) = \mathbb{E}_{x \sim P} \left[ L(h(x), f(x)) \right].
$$
Which distance should we use to compare these distributions?
Simple Analysis

**Proposition:** assume that the loss $L$ is bounded by $M$, then

$$|\mathcal{L}_Q(h, f) - \mathcal{L}_P(h, f)| \leq M l_1(Q, P).$$

**Proof:**

$$|\mathcal{L}_Q(h, f) - \mathcal{L}_P(h, f)| = \left| \mathbb{E}_Q[L((h(x), f(x))] - \mathbb{E}_P[L((h(x), f(x))] \right|$$

$$= \left| \sum_x (Q(x) - P(x)) L((h(x), f(x))] \right|$$

$$\leq M \sum_x \left| Q(x) - P(x) \right|.$$
Example - 0/1 Loss

\[ |\mathcal{L}_Q(h, f) - \mathcal{L}_P(h, f)| = |Q(a) - P(a)| \]
**Definition:**

\[ d_A(Q_1, Q_2) = \sup_{a \in A} |Q_1(a) - Q_2(a)|. \]

where \( A \) is a set of regions or subsets of \( X \) [Devroye et al. (1996), Kifer et al. (2004)], Ben-David et al. (2007), Blitzer et al. (2007)].

For 0/1 loss, the natural choice is the set of all possible disagreement regions:

\[ A = H \Delta H = \{|h' - h| : h, h' \in H\}. \]
Discrepancy Distance

Definition:

\[ \text{disc}(Q_1, Q_2) = \max_{h, h' \in H} \left| \mathcal{L}_{Q_1}(h', h) - \mathcal{L}_{Q_2}(h', h) \right|. \]

- Relationship with discrepancy in combinatorial contexts [Chazelle (2000)].
- \(d_A\) is a special case, 0-1 loss.
- Helps compare distributions for other losses, e.g. hinge loss, \(L_p\) loss.
- Symmetric, verifies triangle inequality, in general not a distance.
Discrepancy - Properties

- **Theorem:** the discrepancy distance can be estimated from finite samples for $H$ with finite VC dimension $d$. For $L_q$ loss, $L_q(y, y') = |y - y'|^q$, for any $\delta > 0$, with probability at least $1 - \delta$,

$$
\text{disc}(P, Q) 
\leq \text{disc}(\hat{P}, \hat{Q}) + O\left(q\sqrt{\frac{d}{m}} + q\sqrt{\frac{d}{n}}\right) + 3M\left(\sqrt{\frac{\log \frac{4}{\delta}}{2m}} + \sqrt{\frac{\log \frac{4}{\delta}}{2n}}\right).
$$
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Theoretical Guarantees

Two types of questions:

• difference between average loss of hypothesis $h$ on $Q$ versus $P$?

• difference of loss between hypothesis $h$ trained on $Q$ and $h'$ trained on $P$. 

Generalization Bound

- **Notation:**
  - $\mathcal{L}_Q(h_Q^*, f) = \min_{h \in H} \mathcal{L}_Q(h, f)$
  - $\mathcal{L}_P(h_P^*, f) = \min_{h \in H} \mathcal{L}_P(h, f)$

- **Theorem:** assume that $L$ obeys the triangle inequality, then the following holds:

$$
\mathcal{L}_P(h, f_P) \leq \mathcal{L}_Q(h, h_Q^*) + \mathcal{L}_P(h_P^*, f_P) + \text{disc}(P, Q) + \mathcal{L}_Q(h_Q^*, h_P^*).
$$
Some Special Cases

- **When** $h^* = h^*_Q = h^*_P$,
  \[
  \mathcal{L}_P(h, f_P) \leq \mathcal{L}_Q(h, h^*) + \mathcal{L}_P(h^*, f_P) + \text{disc}(P, Q).
  \]

- **When** $f_P \in H$ (consistent case),
  \[
  |\mathcal{L}_P(h, f_P) - \mathcal{L}_Q(h, f_P)| \leq \text{disc}(Q, P).
  \]
Kernel-Based Reg. Algorithms

- Algorithms minimizing objective function:

\[ F_Q(h) = \lambda \|h\|^2_K + \hat{R}_Q(h), \]

where \( K \) is a positive definite symmetric kernel, \( \lambda > 0 \) is a trade-off parameter, and \( \hat{R}_Q(h) \) the empirical error of \( h \).

- family of algorithms including SVMs, SVR, kernel ridge regression, etc.
Guarantees for KBR Algorithms

**Theorem:** let $K$ be a positive definite symmetric kernel with $\forall x, K(x, x) \leq \kappa$ and the loss s.t. $L(\cdot, y)$ is $\sigma$-Lipschitz. Assume that $f_P \in H$ and that $f_P$ and $f_Q$ coincide on the training sample. Then, for all $x \in X, y \in Y$,

$$\left| L(h'(x), y) - L(h(x), y) \right| \leq \kappa \sigma \sqrt{\frac{\text{disc}(\hat{P}, \hat{Q})}{\lambda}}.$$
Guarantees for KBR Algorithms

Theorem: same assumptions but $f_P$ and $f_Q$ potentially different on the training sample, $H$ bounded by $M$, and $L$ the square loss; then, for all $x \in X, y \in Y$,

$$|L(h'(x), y) - L(h(x), y)| \leq \frac{2\kappa M}{\lambda} \left( \kappa \delta + \sqrt{\kappa^2 \delta^2 + 4\lambda \text{disc}_L(\hat{P}, \hat{Q})} \right),$$

with $\delta^2 = L_{\hat{Q}}(f_Q(x), f_P(x)) \ll 1$. 
Empirical Discrepancy

- Discrepancy distance $\text{disc}(\hat{P}, \hat{Q})$ critical term in bounds.

- Smaller empirical discrepancy guarantees closeness of pointwise losses of $h'$ and $h$.

- But, can we further reduce the discrepancy?
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Algorithm - Idea

The training sample is given, but we can search for a new empirical distribution $\hat{Q}'$ such that

$$\hat{Q}' = \arg\min_{\hat{Q}' \in Q} \text{disc}(\hat{P}, \hat{Q}')$$

where $Q$ is the set of distributions with support $\text{supp}(\hat{Q})$.

- can be interpreted as reweighting training points.
Case of Halfspaces
Min-Max Problem

Reformulation:

\[ \hat{Q}' = \arg\min_{\hat{Q}' \in Q} \max_{h, h' \in H} |\mathcal{L}_{\hat{P}}(h', h) - \mathcal{L}_{\hat{Q}'}(h', h)|. \]

- game theoretical interpretation.
- gives lower bound:

\[ \max_{h, h' \in H} \min_{\hat{Q}' \in Q} |\mathcal{L}_{\hat{P}}(h', h) - \mathcal{L}_{\hat{Q}'}(h', h)| \leq \min_{\hat{Q}' \in Q} \max_{h, h' \in H} |\mathcal{L}_{\hat{P}}(h', h) - \mathcal{L}_{\hat{Q}'}(h', h)|. \]
Classification - 0/1 Loss

Problem:

\[
\min_{Q'} \max_{a \in H \Delta H} \left| \hat{Q}'(a) - \hat{P}(a) \right|
\]

subject to \( \forall x \in S_Q, \hat{Q}'(x) \geq 0 \wedge \sum_{x \in S_Q} \hat{Q}'(x) = 1. \)
Classification - 0/1 Loss

- Linear program (LP):

\[
\min_{Q'} \delta \\
\text{subject to} \quad \forall a \in H \Delta H, \; \hat{Q}'(a) - \hat{P}(a) \leq \delta \\
\forall a \in H \Delta H, \; \hat{P}(a) - \hat{Q}'(a) \leq \delta \\
\forall x \in S_Q, \; \hat{Q}'(x) \geq 0 \land \sum_{x \in S_Q} \hat{Q}'(x) = 1.
\]

- No. of constraints bounded by shattering coefficient \( \Pi_{H \Delta H}(m_0 + n_0) \).
Algorithm - 1D

A
B
C
D
Regression - L2 Loss

Problem:

\[
\min_{\hat{Q}'} \max_{h,h' \in \mathcal{H}} \left| \underbrace{\mathbb{E}_{\hat{P}}[(h'(x) - h(x))^2] - \mathbb{E}_{\hat{Q}'}[(h'(x) - h(x))^2]}_{\text{empirical risk difference}} \right|
\]
Regression - L2 Loss

- Semi-definite program (SDP): linear hypotheses.

\[
\begin{align*}
\min_{z, \lambda} & \quad \lambda \\
\text{subject to} & \quad \lambda I - M(z) \succeq 0 \\
& \quad \lambda I + M(z) \succeq 0 \\
& \quad 1^T z = 1 \land z \geq 0,
\end{align*}
\]

where the matrix \( M(z) \) is defined by:

\[
M(z) = \sum_{x \in S} \hat{P}(x)xx^T - \sum_{i=1}^{m_0} z_i s_i s_i^T.
\]
Regression - L2 Loss

- **SDP**: generalization to $H$ RKHS for some kernel $K$.

$$\min_{z, \lambda} \lambda$$ subject to

$$\lambda I - M(z) \succeq 0$$
$$\lambda I + M(z) \succeq 0$$
$$1^\top z = 1 \land z \geq 0,$$

with:

$$M(z) = M_0 - \sum_{i=1}^{m_0} z_i M_i$$
$$M_0 = K^{1/2} \text{diag}(P(s_1), \ldots, P(s_{p_0})) K^{1/2}$$
$$M_i = K^{1/2} I_i K^{1/2}.$$
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Experiments

Classification:
- $Q$ and $P$ Gaussians.
- $H$: halfspaces.
- $f$: interval $[-1, +1]$. 

![Graph showing training points with and without minimum discriminator.]
Experiments

Regression:

SDP solved in about 15s using SeDuMi on 3GHz CPU with 2GB memory.
Conclusion

- **Discrepancy distance**: appears as the ‘right’ measure of difference of distributions for adaptation.

- **Theoretical analysis**: generalization bounds and strong guarantees for a large class of algorithms.

- **Algorithm**: discrepancy minimization algorithms for other loss functions, more efficient large-scale algorithms.