Abstract

We present an extensive study of a key problem in online learning where the learner can opt to abstain from making a prediction, at a certain cost. In the adversarial setting, we show how existing online algorithms and guarantees can be adapted to this problem. In the stochastic setting, we first point out a bias problem that limits the straightforward extension of algorithms such as UCB-N to this context. Next, we give a new algorithm, UCB-GT, that exploits historical data and time-varying feedback graphs. We show that this algorithm benefits from more favorable regret guarantees than a natural extension of UCB-N. We further report the results of a series of experiments demonstrating that UCB-GT largely outperforms that extension of UCB-N, as well as other standard baselines.

1. Introduction

We consider an online learning scenario, prevalent in many applications, where the learner is granted the option of abstaining from making a prediction, at a certain cost. For example, in the classification setting, at each round, the learner can choose to make a prediction and incur a standard zero-one misclassification cost, or elect to abstain, in which case she incurs an abstention cost, typically less than one. Abstention can thus represent an attractive option to avoid a higher cost of misclassification. Note, however, that when the learner abstains, she does not receive the true label (correct class), which results in a loss of information.

This scenario of online learning with abstention is relevant to many real-life problems. As an example, consider the scenario where a doctor can choose to make a diagnosis based on the current information available about a patient, or abstain and request further laboratory tests, which can represent both a time delay and a financial cost. In this case, the abstention cost is usually substantially lower than that of a wrong diagnosis. The online model is appropriate since it captures the gradual experience a doctor gains by testing, examining and following new patients.

Another instance of this problem appears in the design of spoken-dialog applications such as those in modern personal assistants. Each time the user asks a question, the assistant can either offer a direct response to the question, at the risk of providing an inaccurate response, or choose to say “I am sorry, I do not understand?”, which results in a longer and thereby more costly dialog requesting the user to reframe his question. Similar online learning problems arise in the context of self-driving cars where, at each instant, the assistant must determine whether to continue steering the car or return the control to the driver. Online learning with abstention also naturally models many problems arising in electronic commerce platforms such as an Ad Exchange, an online platform set up by a publisher where several advertisers bid in order to compete for an ad slot, the abstention cost being the opportunity loss of not bidding for a specific ad slot.

In the batch setting, the problem of learning with abstention has been studied in a number of publications, starting with (Chow, 1957; 1970). Its theoretical aspects have been analyzed by several authors in the last decade. El-Yaniv & Wiener (2010; 2011) studied the trade-off between the coverage and accuracy of classifiers. Bartlett & Wegkamp (2008) introduced a loss function including explicitly an abstention cost and gave a consistency analysis of a surrogate loss that they used to derive an algorithm. More recently, Cortes et al. (2016a;b) presented a comprehensive study of the problem, including an analysis of the properties of a corresponding abstention (or rejection) loss with a series of theoretical guarantees and algorithmic results both for learning with kernel-based hypotheses and for boosting.

This paper presents an extensive study of the problem of online learning with abstention, in both the adversarial and the stochastic settings. We consider the common scenario of prediction with expert advice (Littlestone & Warmuth, 1994) and adopt the same general abstention loss function as in (Cortes et al., 2016a), with each expert formed by a
pair made of a predictor and an abstention function.

A key aspect of the problem we investigate, which makes it distinct from both batch learning with abstention, where labels are known for all training points, and standard online learning (in the full information setting) is the following: if the algorithm abstains from making a prediction for the input point received at a given round, the true label of that point is not revealed. As a result, the loss of the experts that would have instead made a prediction on that point cannot be determined at that round. Thus, we are dealing with an online learning scenario with partial feedback. If the algorithm chooses to predict, then the true label is revealed and the losses of all experts, including abstaining ones, are known. But, if the algorithm elects to abstain, then only the losses of the abstaining experts are known, all of them being equal to the same abstention cost.

As we shall see, our learning problem can be cast as a specific instance of online learning with a feedback graph, a framework introduced by Mannor & Shamir (2011) and later extensively analyzed by several authors (Caron et al., 2012; Alon et al., 2013; 2014; 2015; Kocâk et al., 2014; Neu, 2015; Cohen et al., 2016)). In our context, the feedback graph varies over time, a scenario for which most of the existing algorithms and analyses (specifically, in the stochastic setting) do not readily apply. Our setting is distinct from the KWIK (knows what it knows) framework of Li et al. (2008) and its later extensions, though there are some connections, as discussed in Appendix A.

Our contribution can be summarized as follows. In Section 3, we analyze an adversarial setting both in the case of a finite family of experts and that of an infinite family. We show that the problem of learning with abstention can be cast as that of online learning with a time-varying feedback graph tailored to the problem. In the finite case, we show how ideas from (Alon et al., 2014; 2015) can be extended and combined with this time-varying feedback graph to devise an algorithm, EXP3-ABS, that benefits from favorable guarantees. In turn, EXP3-ABS is used as a subroutine for the infinite case where we show how a surrogate loss function can be carefully designed for the abstention loss, while maintaining the same partial observability. We use the structure of this loss function to extend CONTEXTUAL EXP3 (Cesa-Bianchi et al., 2017) to the abstention scenario and prove regret guarantees for its performance.

In Section 4, we shift our attention to the stochastic setting. Stochastic bandits with a fixed feedback graph have been previously studied by Caron et al. (2012) and Cohen et al. (2016). We first show that an immediate extension of these algorithms to the time-varying graphs in the abstention scenario faces a technical bias problem in the estimation of the expert losses. Next, we characterize a set of feedback graphs that can circumvent this bias problem in the general setting of online learning with feedback graphs. We further design a new algorithm, UCB-GT, whose feedback graph is estimated based on past observations. We prove that the algorithm admits more favorable regret guarantees than the UCB-N algorithm (Caron et al., 2012). Finally, in Section 5 we report the results of several experiments with both artificial and real-world datasets demonstrating that UCB-GT in practice significantly outperforms an unbiased, but limited, extension of UCB-N, as well as a standard bandit baseline, like UCB (Auer et al., 2002a).

2. Learning Problem

Let $\mathcal{X}$ denote the input space (e.g., $\mathcal{X}$ is a bounded subset of $\mathbb{R}^d$). We denote by $\mathcal{H}$ a family of predictors $h: \mathcal{X} \rightarrow \mathbb{R}$, and consider the familiar binary classification problem where the loss $\ell(y, h(x))$ of $h \in \mathcal{H}$ on a labeled pair $(x, y) \in \mathcal{X} \times \{-1, 1\}$ is defined as either the 0/1-loss $1_{yh(x) \leq 0}$, or some Lipschitz variant thereof (see Section 3). In all cases, we assume $\ell(\cdot, \cdot) \in [0, 1]$. We also denote by $\mathcal{R}$ a family of abstention functions $r: \mathcal{X} \rightarrow \mathbb{R}$, with $r(x) \leq 0$ indicating an abstention on $x \in \mathcal{X}$ (or that $x$ is rejected), and $r(x) > 0$ that $x$ is predicted upon (or that $x$ is accepted).

We consider a specific online learning scenario whose regime lies between bandit and full information, sometimes referred to as bandit with side-information (e.g., Mannor & Shamir (2011); Caron et al. (2012); Alon et al. (2013; 2014; 2015; Kocâk et al. (2014); Neu (2015); Cohen et al. (2016)). In our case, the arms are pairs made of a predictor function $h$ and an abstention function $r$ in a given family $\mathcal{E} \subseteq \mathcal{H} \times \mathcal{R}$. We will denote by $\xi = (h_j, r_j)$, $j \in [K]$, the elements of $\mathcal{E}$. In fact, depending on the setting, $K$ may be finite or (countably) infinite. Given $h_j$, one natural choice for the associated abstention function $r_j$ is a confidence-based abstention function of the form $r_j(x) = |h_j(x)| - \theta$, for some threshold $\theta > 0$. Yet, more general pairs $(h_j, r_j)$ can be considered here. This provides an important degree of flexibility in the design of algorithms where abstentions are allowed, as shown in (Cortes et al., 2016a;b). Appendix A presents a concrete example illustrating the benefits of learning with these pair of functions.

The online learning protocol is described as follows. At each round $t \in [T]$, the online algorithm receives an input $x_t \in \mathcal{X}$ and chooses (possibly at random) an arm (henceforth also called “expert” or “pair”) $\xi_t = (h_{t}, r_{t}) \in \mathcal{E}$. If the inequality $r_{t}(x_t) \leq 0$ holds, then the algorithm abstains and incurs as loss an abstention cost $c(x_t) \in [0, 1]$. Otherwise, it predicts based on the sign of $h_{t}(x_t)$, receives the true label $y_t \in \{-1, 1\}$, and incurs the loss $\ell(y_t, h_t(x_t))$. Thus, the overall abstention loss $L$ of expert $\xi = (h, r) \in \mathcal{E}$ on the labeled pair $z = (x, y) \in \mathcal{X} \times \{-1, 1\}$ is defined as...
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Predicting experts $\xi_2$ $\xi_1$ We will consider both an adversarial setting (Section 3), where vertices (see Figure 1 for an example). Observe also that the algorithm at round $t$ observes $L$ that depends on the instance $x_t$ received by the algorithm at round $t$. In our problem, if the learner chooses to predict at round $t$, the set of directed edges at round $t$. The directed edge $\xi_i \rightarrow \xi_j$ is in $E_t$ if the loss of expert $\xi_j \in \mathcal{E}$ is revealed to her. If instead she abstains at round $t$ (i.e., if $r_t(x_t) > 0$), then she only observes $L(\xi_j, z_t)$ of all experts $\xi_j$, since the label $y_t$ is revealed to her. If instead she abstains at round $t$ (i.e., if $r_t(x_t) \leq 0$), then she only observes $L(\xi_j, z_t)$ for those experts $\xi_j$ that are abstaining in that round, that is, the set of $j$ such that $r_j(x_t) \leq 0$, since for all such $\xi_j$, we have $L(\xi_j, z_t) = c$. Notice that in both cases the learner can observe the loss of her own action. Thus, the feedback graph we are operating with is a nearly fully connected directed graph with self-loops, except that it admits only one-way edges from predicting to abstaining vertices (see Figure 1 for an example). Observe also that the feedback graph $G^\text{ABS}_t$ is fully determined by $x_t$.

We will consider both an adversarial setting (Section 3), where no distributional assumption is made about the sequence $z_t = (x_t, y_t)$, $t \in [T]$, and a stochastic setting (Section 4), where $z_t$ is assumed to be drawn i.i.d. from some unknown distribution $\mathcal{D}$ over $\mathcal{X} \times \{\pm 1\}$. For both settings, we measure the performance of an algorithm $A$ by its (pseudo-)regret $R_T(A)$, defined as $R_T(A) = \sup_{\xi \in \mathcal{E}} \mathbb{E}\left[\sum_{t=1}^{T} L(\xi_t, z_t) - \sum_{t=1}^{T} L(\xi, z_t)\right]$, where the expectation is taken both with respect to the algorithm’s choice of actions $I_t$s and, in the stochastic setting, the random draw of the $z_t$s.

In the stochastic setting, we will be mainly concerned with the case where $\mathcal{E}$ is a finite set of experts $\mathcal{E} = \{\xi_1, \ldots, \xi_K\}$. We then denote by $\mu_j$ the expected loss of expert $\xi_j \in \mathcal{E}$, $\mu_j = \mathbb{E}_{z \sim \mathcal{D}}[L(\xi_j, z)]$, by $\mu^*$ the expected loss of the best expert, $\mu^* = \min_{j \in [K]} \mu_j$, and by $\Delta_j$ the loss gap to the best, $\Delta_j = \mu_j - \mu^*$. In the adversarial setting, we will analyze both the finite and infinite expert scenarios. In the infinite case, since $L$ is non-convex in the relevant parameters (Eq. (1)), further care is needed.

3. Adversarial setting

As a warm-up, we start with the adversarial setting with finitely-many experts. Following ideas from Alon et al. (2014; 2015), we design an online algorithm for the abstention scenario by combining standard finite-arm bandit algorithms, like EXP3 (Auer et al., 2003), with the feedback graph $G^\text{ABS}_t$ of Section 2. We call the resulting algorithm EXP3-ABS (EXP3 with abstention). The algorithm is a variant of EXP3 where the importance weighting scheme to achieve unbiased loss estimates is based on the probability of the loss of an expert being observed as opposed to that of an expert being selected — see Appendix B (Algorithm 3). The following guarantee holds for this algorithm.

**Theorem 1** Let EXP3-ABS be run with learning rate $\eta$ over a set of $K$ experts $\xi_1, \ldots, \xi_K$. Then, the algorithm admits the following regret guarantee after $T$ rounds:

$$R_T(\text{EXP3-ABS}) \leq (\log K)/\eta + \eta T(c^2 + 1)/2.$$ 

In particular, if EXP3-ABS is run with $\eta = \sqrt{2 \log K / (c^2 + 1)/T}$, then

$$R_T(\text{EXP3-ABS}) \leq \sqrt{2(c^2 + 1)/T} \log K.$$ 

The proof of this result, as well as all other proofs, is given in the appendix. The dependency of the bound on the number of experts is clearly more favorable than the standard bound for EXP3 ($\sqrt{\log K}$ instead of $\sqrt{K}$). Theorem 1 is in fact reminiscent of what one can achieve using the contextual-bandit algorithm EXP4 (Auer et al., 2002b) run on $K$ experts, each one having two actions.

We now turn our attention to the case of an uncountably infinite $\mathcal{E}$. To model this more general framework, one might be tempted to focus on parametric classes of functions $h$ and $r$, e.g., the family $\mathcal{E}$ of linear functions $\{ (h, r) : h(x) = w^T x, r(x) = |w^T x| - \theta, w \in \mathbb{R}^d, \theta > 0 \}$. introduce some convex surrogate of the abstention loss (1), and work in the parametric space of $(w, \theta)$ through some
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Bandit Convex Optimization technique (e.g., (Hazan, 2016)). Unfortunately, this approach is not easy to put in place, since the surrogate loss not only needs to ensure convexity and some form of calibration, but also the ability for the algorithm to observe the loss of its own action (the self-loops in the graph of Figure 1).

We have been unable to get around this problem by just resorting to convex surrogate losses (and we strongly suspect that it is not possible), and in what follows we instead introduce a surrogate abstention loss which is Lipschitz but not convex. Moreover, we take the more general viewpoint of competing with pairs \((h, r)\) of Lipschitz functions with bounded Lipschitz constant. Let us then consider the version of the abstention loss (1) with \(\ell(y, h(x)) = f_\gamma(-yh(x))\), where \(f_\gamma\) is the 0/1-loss with slope \(1/(2\gamma)\) at the origin, \(f_\gamma(a) = \left(\frac{2a + \gamma}{2\gamma}\right)1_{|a| < \gamma} + 1_{|a| \geq \gamma}\) (see Figure 2 (a)), and the class of experts \(E = \{\xi = (h, r) \mid h, r : \mathcal{X} \subseteq \mathbb{R}^d \rightarrow [-1, 1]\}\). Here, functions \(h\) and \(r\) in the definition of \(E\) are assumed to be \(L_\mathcal{E}\)-Lipschitz with respect to an appropriate distance on \(\mathbb{R}^d\), for some constant \(L_\mathcal{E}\) which determines the size of the family \(\mathcal{E}\).

Using ideas from (Cesa-Bianchi et al., 2017), we present an algorithm that approximates the action space by a finite cover while using the structure of the abstention setting. The crux of the problem is to define a Lipschitz function \(\hat{L}\) that upper bounds the abstention loss while maintaining the same feedback assumptions, namely the feedback graph given in Figure 1. One Lipschitz function \(\hat{L}\) that precisely solves this problem is the following:

\[
\hat{L}(\xi, z) = \begin{cases} 
    c & \text{if } r(x) \leq -\gamma \\
    1 + \left(\frac{1+c}{\gamma}\right) r(x) & \text{if } r(x) \in (-\gamma, 0) \\
    1 - \left(1 - f_\gamma(-yh(x))\right) r(x) & \text{if } r(x) \in [0, \gamma) \\
    f_\gamma(-yh(x)) & \text{if } r(x) \geq \gamma,
\end{cases}
\]

for \(\gamma \in (0, 1)\). \(\hat{L}(\xi, z)\) is plotted in Figure 2(b). Notice that this function is consistent with the feedback requirements of Section 2: \(r_t(x_t) \leq 0\) implies that \(\hat{L}((h(x_t), r(x_t)), z_t)\) is known to the algorithm (i.e., is independent of \(y_t\)) for all \((h, r) \in E\) such that \(r(x_t) \leq 0\), while \(r_t(x_t) > 0\) gives complete knowledge of \(\hat{L}((h(x_t), r(x_t)), z_t)\) for all \((h, r) \in E\), since \(y_t\) is observed.

We can then adapt the machinery from (Cesa-Bianchi et al., 2017) so as to apply a contextual version of EXP3-ABS to the sequence of losses \(\hat{L}(\xi, z_t), t \in [T]\). The algorithm adaptively covers \(\mathcal{X}\) with balls of a fixed radius \(\varepsilon\), each ball hosting an instance of EXP3-ABS. We call this algorithm \textsc{Contexp3-ABS} – see Appendix B.2 for details.

**Theorem 2** Consider the abstention loss

\[
L(\xi, z) = f_\gamma(-yh(x))1_{r(x) > 0} + c1_{r(x) \leq 0},
\]

and let \(\xi^* = (h^*, r^*) = \arg\min_{\xi \in \mathcal{E}} \sum_{t=1}^T L(\xi, z_t)\), with \(\mathcal{E} = \{(h, r)\} \) made of pairs of Lipschitz functions as described above. If \textsc{Contexp3-ABS} is run with parameter \(\varepsilon \simeq \frac{T}{\mathcal{E}^2} \frac{\gamma}{\gamma + \varepsilon}\) and an appropriate learning rate (see Appendix B), then, it admits the following regret guarantee:

\[
R_T(\textsc{Contexp3-ABS}) \leq \tilde{O} \left(T \frac{\gamma}{\gamma + \varepsilon} \gamma^{-\gamma} + M_\gamma^2(\gamma)\right),
\]

where \(M_\gamma^2(\gamma)\) is the number of \(x_t\) such that \(|r^*(x_t)| \leq \gamma\).

In the above, \(\tilde{O}\) hides constant and \(\ln(T)\) factors, while \(\simeq\) disregards constants like \(L_\mathcal{E}\), and various log factors. \textsc{Contexp3-ABS} is also computationally efficient, thereby providing a compelling solution to the infinite armed case of online learning with abstention.

**4. Stochastic setting**

We now turn to studying the stochastic setting. As pointed out in Section 2, the problem can be cast as an instance of online learning with time-varying feedback graphs \(G_t^\text{ABS}\). Thus, a natural method for tackling the problem would be to extend existing algorithms designed for the stochastic setting with feedback graphs to our abstention scenario (Cohen et al., 2016; Caron et al., 2012). We cannot benefit from the algorithm of Cohen et al. (2016) in our scenario. This is because at the heart of its design and theoretical guarantees lies the assumption that the graphs and losses are independent. The dependency of the feedback graphs on the observations \(z_t\), which also define the losses, is precisely a property that we wish to exploit in our scenario.

An alternative is to extend the UCB-N algorithm of Caron et al. (2012), for which the authors provide gap-based regret guarantees. This algorithm is defined for a stochastic setting with an undirected feedback graph that is fixed over time. The algorithm can be straightforwardly extended to the case of directed time-varying feedback graphs (see Algorithm 1). We will denote that extension by UCB-NT to explicitly differentiate it from UCB-N. Let \(N_t(j)\) denote the set of out-neighbors of vertex \(\xi_j\) in the directed graph at time \(t\), i.e., the set of vertices \(\xi_k\) destinations of an edge from \(\xi_j\). Then, as with UCB-N, the algorithm updates, at
each round $t$, the upper-confidence bound of every expert for which a feedback is received (those in $N_t(I_t)$), as opposed to updating only the upper-confidence bound of the expert selected, as in the standard UCB of Auer et al. (2002a).

In the context of learning with abstention, the natural feedback graph $G_t^{\text{ABS}}$ at time $t$ depends on the observation $x_t$ and varies over time. Can we extend the regret guarantees of Caron et al. (2012) to UCB-NT with such graphs? We will show in Section 4.1 that vanishing regret guarantees do not hold for UCB-NT run with graphs $G_t^{\text{ABS}}$. This is because of a fundamental estimation bias problem that arises when the graph at time $t$ depends on the observation $x_t$. This issue affects more generally any natural method using the $G_t^{\text{ABS}}$ graphs. Nevertheless, we will show in Section 4.2 that UCB-NT does benefit from favorable guarantees, provided the feedback graph $G_t^{\text{ABS}}$ it uses at round $t$ is replaced by one that only depends on events up to time $t - 1$.

### 4.1. Bias problem

Assume there are two experts: $\xi_1$ (red) and $\xi_2$ (blue) with $\mu_2 < \mu_1$ and $X = [0, 1]$ (see Figure 3). For $x > \frac{1}{2}$, the red expert $\xi_1$ is abstaining and incurring a loss $c$, whereas the blue expert is never abstaining. Assume that the probability mass is quasi-uniform over the interval $[0, 1]$ but with slightly more mass over the region $x < \frac{1}{2}$. The algorithm may start out by observing points in this region. Here, both experts accept and the algorithm obtains error estimates corresponding to the solid red and blue lines for $x < \frac{1}{2}$. When the algorithm observes a point $x > \frac{1}{2}$, it naturally selects the red abstaining expert since it admits a better current estimated loss. However, for $x > \frac{1}{2}$, the red expert is worse than the blue expert $\xi_2$. Furthermore, it is abstaining and thus providing no updates for expert $\xi_2$ (which is instead predicting). Hence, the algorithm continues to maintain an estimate of $\xi_2$’s loss at the level of the blue solid line indicated for $x < \frac{1}{2}$, it then continues to select the red expert for all $x$s and incurs a high regret.\(^1\)

This simple example shows that, unlike the adversarial scenario (Section 3), $G_t^{\text{ABS}}$, here, cannot depend on the input $x_t$, and that, in general, the indiscriminate use of feedback graphs may result in biased loss observations. On the other hand, we know that if we were to avoid using feedback graphs at all (which is always possible using UCB), we would always be able to define unbiased loss estimates.

A natural question is then: can we construct time-varying feedback graphs that lead to unbiased loss observations? In the next section, we show how to design such a sequence of auxiliary feedback graphs, which in turn allows us to then extend UCB-NT to the setting of time-varying feedback graphs for general loss functions. Under this assumption, we can achieve unbiased empirical estimates of the average losses $\mu_j$ of the experts, which will allow us to apply standard concentration bounds in the proof of this algorithm.

### 4.2. Time-varying graphs for UCB-NT

We now show that UCB-NT benefits from favorable guarantees, so long as the feedback graph $G_t^{\text{ABS}}$ it uses at time $t$ depends only on events up to time $t - 1$. This extension works for general bounded losses and does not only apply to our specific abstention loss $L$.

So, let us assume that the feedback graph in round $t$ (and the associated out-neighborhoods $N_t(.)$) in Algorithm 1 only depends on the observed losses $L(\xi_t, z_s)$ and inputs $x_s$, for $s = 1, \ldots, t - 1$, and $i \in [K]$, and let us denote this feedback graph by $G_t$, so as not to get confused with $G_t^{\text{ABS}}$. Under this assumption, we can derive strong regret guarantees for UCB-NT with time-varying graphs. Our guarantees are expressed in terms of the best sequence of admissible $p$-partitionings of graphs $G_t$. For $p \in [K]$, we say that $(C_t,k)_{k \in [p]}$ is an admissible $p$-partitioning of $G_t = (V,E_t)$ if $V = \{\xi_j : j \in [K]\}$ is the union of $p$ (disjoint) components $C_t,k$, that is $V = \bigcup_{k \in [p]} C_t,k$, and all vertices within a component $C_t,k$ are neighbors in $G_t$ with reciprocal edges: if $\xi_i$ and $\xi_j$ are in $C_t,k$, then we have $\xi_i \in N_t(\xi_j)$ and $\xi_j \in N_t(\xi_i)$. Note that admissible $p$-partitionings are typically not unique since two neighbor vertices $\xi_i$ and $\xi_j$ may be placed in the same component or not. We denote by $S_p$ the set of all sequences $((C_1,k)_{k \in [p]}, \ldots, (C_T,k)_{k \in [p]})$ of admissible $p$-partitionings $(C_t,k)_{k \in [p]}$ of graphs $G_t$. Moreover, for any sequence $((C_1,k)_{k \in [p]}, \ldots, (C_T,k)_{k \in [p]})$ in $S_p$, we denote by $C_k$ the union of the components indexed by $k$ over all

![Figure 3: Illustration of the bias problem.](image-url)
The regret of Theorem 3 can be interpreted as an extension of Theorem 2. Assume that, for all $t \in [T]$, the feedback graph $G_t$ depends only on information up to time $t - 1$. Then, the regret of UCB-NT is bounded as follows:

$$O \left( \mathbb{E} \left[ \min_{p, S_p} \sum_{k \in [p]} \frac{\max_{j \in e_k} \Delta_j}{\min_{j \in e_k} \Delta_j} \log(T) + K \right] \right).$$

For a sequence $S_p$ made up of the same partition $(e_{1,k})_{k \in [p]}$ repeated $T$ times, the theorem gives a bound on the regret based on this fixed partition, as it is the sum of $p$ components, one per cluster $e_{1,k}$ in the partition. The minimum over $(p, S_p)$ then simply chooses the number of clusters $p$ and the partitioning of $V$ into $p$ clusters having the smallest regret.

Theorem 3 can be interpreted as an extension of Theorem 2 in Caron et al. (2012) to time-varying feedback graphs. Its proof involves showing that the use of feedback graphs $G_t$ that depend only on information up to $t - 1$ can result in unbiased loss estimates, and it also uses the newly defined notion of admissible $p$-partitionings to derive a time-varying bound that leverages the shared updates from the graph.

Moreover, the bound illustrates that if the feedback graphs in a problem admit a $p$-partitioning for some small $p \ll K$ (e.g., if the feedback graphs can be decomposed into a small number of components that are approximately fixed across time) for which $\max_{j \in e_k} \Delta_j \approx \min_{j \in e_k} \Delta_j$, then this bound can be up to a factor $\frac{p}{K}$ tighter than the bound guaranteed by the standard UCB algorithm. Moreover, this regret guarantee is always more favorable than that of the standard UCB since the (trivial) $K$-partitioning that splits $V$ into $K$ singletons for all $t$ is an admissible $K$-partitioning for all $G_t$'s. Furthermore, note that by construction, all vertices within the same component of an admissible $p$-partitioning are connected to one another. Thus, if the feedback graph is fixed throughout all rounds and we interpret the doubly-directed edges as undirected edges of an undirected graph $\overline{G}$, we straightforwardly obtain the following result, which is comparable to Theorem 2 in Caron et al. (2012).

Corollary 1 If the feedback graph $G_t = G$ is fixed over time, then the guarantee of Theorem 3 is upper-bounded by:

$$O \left( \min_{C} \sum_{C e} \frac{\max_{j \in e} \Delta_j}{\min_{j \in e} \Delta_j} \log(T) + K \right),$$

the outer minimum being over all clique coverings $C$ of $\overline{G}$.

Caron et al. (2012) present matching lower bounds for the case of stochastic bandits with a fixed feedback graph. Since we can again design abstention scenarios with fixed feedback graphs, these bounds carry over to our setting.

Now, how can we use the results of this section to design an algorithm for the abstention scenario? The natural feedback graphs we discussed in Section 3 are no longer applicable since $G_t^{ABS}$ depends on $x_t$. Nevertheless, we will present two solutions to this problem. In Section 4.3, we present a solution with a fixed graph $G$ that closely captures the problem of learning with abstention. Next, in Section 4.4, we will show how to define and leverage a time-varying graph $G_t$ that is estimated based on past observations.

4.3. UCB-N with the subset feedback graph

In this section, we define a subset feedback graph, $G^{SUB}$, that captures the most informative feedback in the problem of learning with abstention and yet is safe in the sense that it does not depend on $x_t$. The definition of the graph is based on the following simple observation: if the abstention region associated with $\xi_i$ is a subset of that of $\xi_j$, then, if $\xi_i$ is selected at some round $t$ and is abstaining, so is $\xi_j$. For an example, see $\xi_i$ and $\xi_j$ in Figure 4 (top). Crucially, this implication holds regardless of the particular input point $x_t$ received in the region of abstention of $\xi_i$. Thus, the set of vertices of $G^{SUB}$ is $\mathcal{E}$, and $G^{SUB}$ admits an edge from $\xi_i$ to $\xi_j$, iff $\{x \in \mathcal{X} : r_t(x) = 0\} \subseteq \{x \in \mathcal{X} : r_t(x) = -1\}$. Since $G^{SUB}$ does not vary with time, it trivially verifies the condition of the previous section. Thus, UCB-NT run with $G^{SUB}$ admits the regret guarantees of Theorem 3, where the admissible $p$-partitionings are those of fixed graph $G^{SUB}$.

The example of Section 4.1 illustrated a bias problem in a special case where the feedback graphs $G_t$ were not subgraphs of $G^{SUB}$. The following result shows more generally that feedback graphs not included in $G^{SUB}$ may result in catastrophic regret behavior.

Proposition 1 Assume that UCB-NT is run with feedback graphs $G_t$ that are not subsets of $G^{SUB}$. Then, there exists a family of predictors $\mathcal{H}$, a Lipschitz loss function $\ell$ in (1), and a distribution $\mathcal{D}$ over $z_t$s for which UCB-NT incurs linear regret with arbitrarily high probability.

The proof of the proposition is given in Appendix C.3. In view of this result, no fixed feedback graph for UCB-NT can be more informative than $G^{SUB}$. But how can we leverage past observations (up to time $t - 1$) to derive a feedback graph that would be more informative than the simple subset graph $G^{SUB}$? The next section provides a solution based on feedback graphs estimated based on past observations and a new algorithm.

4.4. UCB-GT algorithm

We seek graphs $G_t$ that admit $G^{SUB}$ as a subgraph. We will show how certain types of edges can be safely added to $G^{SUB}$ based on past observations. This leads to a new algorithm, UCB-GT (UCB with estimated time-varying graph), whose pseudocode is given in Algorithm 2. As illustrated by Figure 4, the key idea of UCB-GT is to augment...
Algorithm 2: UCB-GT

for $t \geq 1$ do
  RECVIE$(x_t)$;
  $\xi_t \leftarrow \arg \min_{\xi \in \mathcal{E}} \{ \hat{\mu}_{i,t-1} - S_t, i - t - 1 \}$,
  where $S_{i,t-1}$ is as in Algorithm 1;
  for $\xi_t \in \mathcal{E}$ do
    if $\hat{\mu}_{i,t} - r_i, i - t - 1 \leq \gamma_{i,t-1}$ then $Q_i, t = Q_i, t - 1 + 1$;
    if $r_i, i(t) \leq 0 \land r_i, i(x) > 0$ then
      $\hat{\mu}_{i,t} \leftarrow \left( 1 - \frac{1}{Q_i, t} \right) \hat{\mu}_{i,t-1}$;
    else $Q_i, t \leftarrow Q_i, t - 1$,
  end for
end for

\( G_{\text{SUB}} \) with edges from $\xi_j$ to $\xi_i$, where the subset property \( \{ x: r_j(x) \leq 0 \} \subseteq \{ x: r_i(x) \leq 0 \} \) may not hold, but where the implication \( r_j(x) \leq 0 \Rightarrow r_i(x) \leq 0 \) holds with high probability over the choice of $x \in X$, that is, the region \( \{ x: r_j(x) \leq 0 \land r_i(x) > 0 \} \) admits low probability. Of course, adding such an edge $\xi_j \rightarrow \xi_i$ can cause the estimation bias of Section 4.1. But, if we restrict ourselves to cases where $p_{j,i} = \mathbb{P}[r_j(x) \leq 0 \land r_i(x) > 0]$ is upper bounded by some carefully chosen quantity that changes over rounds, the effect of this bias will be limited. In reverse, as illustrated in Figure 4, the resulting feedback graph can be substantially more beneficial since it may have many more edges than $G_{\text{SUB}}$, hence leading to more frequent updates of the experts’ losses and more favorable regret guarantees. This benefit is further corroborated by our experimental results (Section 5).

Since we do not have access to $p_{j,i}$, we use instead empirical estimates \( \hat{p}_{j,i} = \frac{1}{t} \sum_{s=1}^{t-1} 1_{r_j(x) \leq 0 \land r_i(x) > 0} \). At time $t$, if expert $\xi_j$ is selected, we update expert $\xi_i$ if the condition \( \hat{p}_{j,i} \leq \hat{\gamma}_{j,i} \) holds with \( \hat{\gamma}_{j,i} = \sqrt{5} Q_{i,t} (1 - 1/(K - 1) (t - 1)) \). If the expert $\xi_j$ chosen abstains while expert $\xi_j$ predicts and satisfies \( \hat{p}_{j,i} \leq \hat{\gamma}_{j,i} \), then we do not have access to the true label $y_t$. In that case, we update optimistically our empirical estimate as if the expert had lost 0 at that round (Step (*) in Alg. 2).

The feedback graph $G_t$ just described can be defined via the out-neighborhood of vertex $\xi_j$: \( N_j{(t)} = \{ \xi_i \in \mathcal{E} : \hat{p}_{j,i} \leq \hat{\gamma}_{j,i} \} \). Let \( \delta_p = (G_k, k) \in \mathcal{T}, k \in [p] \) denote any sequence of admissible $p$-partitionings of these feedback graphs, then the following regret guarantee holds for UCB-GT.

Theorem 4 For any $t \in [T]$, let the feedback graph $G_t$ be defined by the out-neighborhood $N_j{(t)} = \{ \xi_i \in \mathcal{E} : \hat{p}_{j,i} \leq \hat{\gamma}_{j,i} \}$. Then, the regret of UCB-GT is bounded as follows:

\[
O \left( \mathbb{E} \left[ \min_{p, \delta_p} \sum_{k \in [p]} \max_{i \in \delta_p} \min_{\xi_j} \frac{\Delta_j}{\Delta_j^2} \log(T) + K \right] \right).
\]

Figure 4: The top row shows three experts $\xi_i$, $\xi_j$, and $\xi_k$, on a one-dimensional input space marked by their prediction and abstention regions. Below each region, the time-varying graph $G_t^{\text{ABS}}$ is shown. To avoid the bias problem affecting the graphs $G_t^{\text{ABS}}$, one option is to use $G_{\text{SUB}}$. Yet, as illustrated, $G_{\text{SUB}}$ is minimal and in this example admits only one edge (excluding self-loops). Thus, a better option is to use the time-varying graphs of UCB-GT since they are richer and more informative. For these graphs, an edge is added from $\xi_j$ to $\xi_i$ if the probability of the region where $\xi_j$ is abstaining but $\xi_i$ is predicting is (estimated to be) small.

Since the graph $G_t$ of UCB-GT has more edges than $G_{\text{SUB}}$, it admits more admissible partitionings than $G_{\text{SUB}}$, which leads to a more favorable guarantee than that of UCB-NT run with $G_{\text{SUB}}$. The proof of this result differs from the standard UCB analysis and that of Theorem 3 in that it involves showing that the UCB-GT algorithm can adequately control the amount of bias introduced by the skewed loss estimates. The experiments in the next section provide an empirical validation of this theoretical comparison.

5. Experiments

In this section, we report the results of several experiments on ten datasets comparing UCB-GT, UCB-NT with feedback graph $G_{\text{SUB}}$, vanilla UCB (with no sharing information across experts), as well as Full-Supervision, FS. FS is an algorithm that at each round chooses the expert $\xi_j$ with the smallest abstention loss so far, $\hat{\mu}_{j,t-1}$, and even if this expert abstains, the algorithm receives the true label and can update the empirical abstention loss estimates for all experts. FS reflects an unrealistic and overly optimistic scenario that clearly falls outside the abstention setting, but it provides an upper bound for the best performance we may hope for.

We used the following eight datasets from the UCI data repository: HIGGS, phishing, ijcnn, covtype, eye, skin, cod-rna, and guide. We also used the CIFAR dataset from (Krizhevsky et al., 2009), where we extracted the first twenty-five principal components and used their
Figure 5: From the left, we show graphs of the average regret $R_t(\cdot)/t$, fraction of points the chosen expert abstained on, and the number of edges of the feedback graph as a function of $t$ (log-scale) for UCB-GT, UCB-NT, UCB, and FS. Top row is the results for cod-rna for cost $c = 0.2$ and bottom row is the guide for cost $c = 0.1$. More results are in Appendix D.

projections as features, and a synthetic dataset of points drawn according to the uniform distribution in $[-1, 1]^2$. For each dataset, we generated a total of $K = 2,100$ experts and all the algorithms were tested for a total of $T = 10,000$ rounds. The experts, $\xi = (h, r)$, were chosen in the following way. The predictors $h$ are hyperplanes centered at the origin whose normal vector in $\mathbb{R}^d$ is drawn randomly from the Gaussian distribution, $\mathcal{N}(0, 1)_d$, where $d$ is the dimension of the feature space of the dataset. The abstention functions $r$ are concentric annuli around the origin with radii in $(0, \sqrt{d}, 2\sqrt{d}, \ldots, \sqrt{d})$. For each dataset, we generated 100 predictors and each predictor $h$ is paired with the 21 abstention functions $r$. For a fixed set of experts, we first calculated the regret by averaging over five random draws of the data, where the best-in-class expert was determined in hindsight as the one with the minimum average cumulative abstention loss. We then repeated this experiment five times over different sets of experts and averaged the results. We report these results for $c \in \{0.1, 0.2, 0.3\}$.

Figure 5 shows the averaged regret $R_t(\cdot)/t$ with standard deviations across the five repetitions for the different algorithms as a function of $t \in [T]$ for two datasets. In Appendix D, we present plots of the regret for all ten datasets. These results show that UCB-GT outperforms both UCB-NT and UCB on all datasets for all abstention cost values. Remarkably, UCB-GT’s performance is close to that of FS for most datasets, thereby implying that UCB-GT attains almost the best regret that we could hope for. We also find that UCB-NT performs better than the vanilla UCB.

Figure 5 also illustrates the fraction of points in which the chosen expert abstains, as well as the number of edges in the feedback graph as a function of rounds. We only plot the number of edges of UCB-GT since that is the only graph that varies with time. For both experiments depicted and in general for the rest of the datasets, the number of edges for UCB-GT is between 1 million to 3 million, which is at least a factor of 5 more than for UCB-NT, where the number of edges we observed are of the order 200,000. FS enjoys the full information property and the number of edges is fixed at 4 million (complete graph). The increased information sharing of UCB-GT is clearly a strong contributing factor to the algorithm’s improvement in regret relative to UCB-NT. In general, we find that, provided that the estimation bias is controlled, the higher is the number of edges, the smaller the regret. Regarding the value of the cost $c$, as expected, we observe that the fraction of points that the chosen expert abstains on always decreases as $c$ increases, but also that that fraction depends on the dataset and the experts used.

Finally, Appendix D includes more experiments for different aspects of the problem. In particular, we tested how the number of experts or a different choice of experts (confidence-based experts) affected the results. We also experimented with some extreme abstention costs and, as expected, found the fraction of abstained points to be large for $c = 0.001$ and small for $c = 0.9$. In all of these additional experiments, UCB-GT outperformed UCB-NT.

6. Conclusion

We presented a comprehensive analysis of the novel setting of online learning with abstention, including algorithms with favorable guarantees both in the stochastic and adversarial scenarios, and extensive experiments demonstrating the performance of UCB-GT in practice. Our algorithms and analysis can be straightforwardly extended to similar problems, including the multi-class and regression settings, as well as other related scenarios, such as online learning with budget constraints. A key idea behind the design of our algorithms in the stochastic setting is to leverage the stochastic sequence of feedback graphs. This idea can perhaps be generalized and applied to other problems where time-varying feedback graphs naturally appear.
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References


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A. Further Related Work

Learning with abstention is a useful paradigm in applications where the cost of misclassifying a point is high. More concretely, suppose the cost of abstention $c$ is less than $1/2$ and consider the set of points along the real line illustrated in Figure 6 where $+$ and $-$ indicate their labels. The best threshold classifier is the hypothesis given by threshold $\theta$, since it correctly classifies points to the right of $\eta$, with an expected loss of $(1/2) \mathbb{P}[x \leq \eta]$. On the other hand, the best abstention pair $(h, r)$ would abstain on the region left of $\eta$ and correctly classify the rest, with an expected loss of $c \mathbb{P}(x \leq \eta)$. Since $c < 1/2$, the abstention pair always admits a better loss then the best threshold classifier.

Within the online learning literature, work related to our scenario includes the KWIK (knows what it knows) framework of Li et al. (2008) in which the learning algorithm is required to make only correct predictions but admits the option of abstaining from making a prediction. The objective is then to learn a concept exactly with the fewest number of abstentions. If in our framework we received the label at every round, KWIK could be seen as a special case of our framework for online learning with abstention with an infinite misclassification cost and some finite abstention cost. A relaxed version of the KWIK framework was introduced and analyzed by Sayedi et al. (2010) where a fixed number $k$ of incorrect predictions are allowed with a learning algorithm related to the solution of the 'mega-egg game puzzle'. A theoretical analysis of learning in this framework was also recently given by Zhang & Chaudhuri (2016). Our framework does not strictly cover this relaxed framework. However, for some choices of the misclassification cost depending on the horizon, the framework is very close to ours. The analysis in these frameworks was given in terms of mistake bounds since the problem is assumed to be realizable. We will not restrict ourselves to realizable problems and, instead, will provide regret guarantees.

B. Additional material for the adversarial setting

We first present the pseudocode and proofs for the finite arm setting and next analyze the infinite arm setting.

B.1. Finite arm setting

Algorithm 3 contains the pseudocode for $\text{EXP}^{3}$-ABS, an algorithm for online learning with abstention under an adversarial data model that guarantees small regret. The algorithm itself is a simple adaptation of the ideas in (Alon et al., 2014; 2015), where we incorporate the side information that the loss of an abstaining arm is always observed, while the loss of a predicting arm is observed only if the algorithm actually plays a predicting arm. In the pseudocode and in the proof that follows, $L_t(\xi_j)$ is a shorthand for $L(\xi_j, (x_t, y_t))$.

Proof of Theorem 1.

Proof. By applying the standard regret bound of Hedge (e.g., (Bubeck & Cesa-Bianchi, 2012)) to distributions $q_1, \ldots, q_T$ generated by $\text{EXP}^{3}$-ABS and to the non-negative loss estimates $\hat{L}_t(\xi_j)$, the following holds:

$$
\mathbb{E} \left[ \sum_{t=1}^{T} \sum_{\xi_j \in \mathcal{E}} q_t(\xi_j) \mathbb{E} \left[ \hat{L}_t(\xi_j) \right] - \sum_{t=1}^{T} \mathbb{E} \left[ \hat{L}_t(\xi^*) \right] \right] \leq \frac{\log K}{\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \mathbb{E} \left[ \sum_{\xi_j \in \mathcal{E}} q_t(\xi_j) \mathbb{E} \left[ \hat{L}_t(\xi_j)^2 \right] \right],
$$

for any fixed $\xi^* \in \mathcal{E}$. Using the fact that $\mathbb{E} \left[ \hat{L}_t(\xi_j) \right] = L_t(\xi_j)$ and $\mathbb{E} \left[ L_t(\xi_j)^2 \right] = \frac{L_t(\xi_j)^2}{P_t(\xi_j)}$, we can write

$$
\mathbb{E} \left[ \sum_{t=1}^{T} \sum_{\xi_j \in \mathcal{E}} q_t(\xi_j) L_t(\xi_j) - \sum_{t=1}^{T} L_t(\xi^*) \right] \leq \frac{\log K}{\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \mathbb{E} \left[ \sum_{\xi_j \in \mathcal{E}} q_t(\xi_j) P_t(\xi_j) L_t(\xi_j)^2 \right].
$$

For each $t$, we can split the nodes $V$ of $G_t^{\text{ABS}}$ into the two subsets $V_{\text{abs},t}$ and $V_{\text{acc},t}$ where if a node $\xi_j$ is abstaining at time $t$
Algorithm 3: EXP3-ABS

**Input** Set of experts $\mathcal{E} = \{\xi_1, \ldots, \xi_K\}$; learning rate $\eta > 0$.

**Init:** $q_1$ is the uniform distribution over $\mathcal{E}$.

for $t \leftarrow 1, 2, \ldots$ do

1. RECEIVE($x_t$);
2. $\xi_t \leftarrow \text{SAMPLE}(q_t)$;
3. if $r_t(x_t) > 0$ then RECEIVE($y_t$);

end for

Algorithm 4: CONEXP3-ABS

**Input** Ball radius $\varepsilon > 0$, $\varepsilon$-covering $y_\varepsilon$ of $y$ such that $|y_\varepsilon| \leq C_y \varepsilon^{-2}$.

for $t \leftarrow 1, 2, \ldots$ do

1. RECEIVE($x_t$);
2. If $x_t$ does not belong to any existing ball, create new ball of radius $\varepsilon$ centered on $x_t$, and allocate fresh instance of EXP3-ABS;
3. Let “Active EXP3-ABS” be the instance allocated to the existing ball whose center $x_s$ is closest to $x_t$;
4. Draw action $\xi_t \in y_\varepsilon$ using Active EXP3-ABS;
5. Get loss feedback associated with $\xi_t$, and use it to update state of “Active EXP3-ABS”.

end for

then $\xi_j \in V_{\text{abs}, t}$, and otherwise $\xi_j \in V_{\text{acc}, t}$. Thus, for any round $t$, we can write

$$
\sum_{\xi_j \in \mathcal{E}} q_t(\xi_j) L_t(\xi_j)^2 = \sum_{\xi_j \in V_{\text{abs}, t}} q_t(\xi_j) L_t(\xi_j)^2 + \sum_{\xi_j \in V_{\text{acc}, t}} q_t(\xi_j) L_t(\xi_j)^2 \\
\leq \sum_{\xi_j \in V_{\text{abs}, t}} q_t(\xi_j) c^2 + \sum_{\xi_j \in V_{\text{acc}, t}} q_t(\xi_j) \frac{L_t(\xi_j)}{P_t(\xi_j)} \\
\leq c^2 + 1.
$$

The first inequality holds since if $\xi_j$ is an abstaining expert at time $t$, we know that $L_t(\xi_j) = c$ and $P_t(\xi_j) = 1$, while for the accepting experts we know that $L_t(\xi_j) \leq 1$ anyway. The second inequality holds because if $\xi_j$ is an accepting expert, we have $P_t(\xi_j) = \sum_{\xi_j \in V_{\text{acc}, t}} q_t(\xi_j)$. Combining this inequality with (2) concludes the proof. □

B.2. Infinite arm setting

Here, the input space $X$ is assumed to be totally bounded, so that there exists a constant $C_X > 0$ such that, for all $0 < \varepsilon \leq 1$, $X$ can be covered with at most $C_X \varepsilon^{-d}$ balls of radius $\varepsilon$. Let $\bar{y}$ be a shorthand for $[-1, 1]^2$, the range space of the pairs $(h, r)$. An $\varepsilon$-covering $y_\varepsilon$ of $\bar{y}$ with respect to the Euclidean distance on $\bar{y}$ has size $K_\varepsilon \leq C_y \varepsilon^{-2}$ for some constant $C_y$.

The online learning scenario for the loss $\bar{L}$ under the abstention setting’s feedback graphs is as follows. Given an unknown sequence $z_1, z_2, \ldots$ of pairs $z_t = (x_t, y_t) \in X \times \{\pm 1\}$, for every round $t = 1, 2, \ldots$.
1. The environment reveals input $x_t \in \mathcal{X}$;
2. The learner selects an action $\xi_{t_\epsilon} \in \mathcal{Y}$ and incurs loss $\tilde{L}(\xi_{t_\epsilon}, z_t)$;
3. The learner obtains feedback from the environment.

Our algorithm is described as Algorithm 4. The algorithm essentially works as follows. At each round $t$, if a new incoming input $x_t \in \mathcal{X}$ is not contained in any existing ball generated so far, then a new ball centered at $x_t$ is created, and a new instance of EXP3-ABS is allocated to handle $x_t$. Otherwise, the EXP3-ABS instance associated with the closest input so far is used. Each allocated EXP3-ABS instance operates on the discretized action space $\mathcal{Y}_\epsilon$.

Consider the function

$$\tilde{L}(a, r) = \begin{cases} 
c & \text{if } r \leq -\gamma \\
1 + \left(\frac{1-r}{\gamma}\right) r & \text{if } r \in (-\gamma, 0) \\
1 - \left(1-f_\gamma(-a)\right) r & \text{if } r \in [0, \gamma) \\
f_\gamma(-a) & \text{if } r \geq \gamma,
\end{cases}$$

where $f_\gamma$ is the Lipschitz variant of the 0/1-loss mentioned in Section 3 of the main text (Figure 2 (a)). For any fixed $a$, the function $\tilde{L}(a, r)$ is $1/\gamma$-Lipschitz when viewed as a function of $r$, and is $1/(2\gamma)$-Lipschitz for any fixed $r$ when viewed as a function of $a$. Hence

$$|\tilde{L}(a, r) - \tilde{L}(a', r')| \leq |\tilde{L}(a, r) - \tilde{L}(a, r')| + |\tilde{L}(a, r') - \tilde{L}(a', r')|$$

$$\leq \frac{1}{\gamma} |r - r'| + \frac{1}{2\gamma} |a - a'|$$

$$\leq \sqrt{\frac{1}{\gamma^2} + \frac{1}{4\gamma^2}} \sqrt{(a - a')^2 + (r - r')^2}$$

$$< \frac{1}{\gamma} \sqrt{(a - a')^2 + (r - r')^2},$$

so that $\tilde{L}$ is $\frac{2}{\gamma}$-Lipschitz w.r.t. the Euclidean distance on $\mathcal{Y}$. Furthermore, a quick comparison to the abstention loss

$$L(a, r) = f_\gamma(a)\mathbb{1}_{r > 0} + c\mathbb{1}_{r \leq 0}$$

reveals that (recall Figure 2 (b) in the main text):

- $\tilde{L}$ is an upper bound on $L$, i.e.,
  $$\tilde{L}(a, r) \geq L(a, r), \quad \forall (a, r) \in \mathcal{Y};$$

- $\tilde{L}$ approximates $L$ in that
  $$\tilde{L}(a, r) = L(a, r), \quad \forall (a, r) \in \mathcal{Y} : |r| \geq \gamma .$$

With the above properties of $\tilde{L}$ at hand, we are ready to prove Theorem 2.

**Proof of Theorem 2.**

**Proof.** On each ball $B \subseteq \mathcal{X}$ that CONEXP3-ABS allocates during its online execution, Theorem 1 supplies the following regret guarantee for the associated instance of EXP3-ABS:

$$\frac{\log K_{\epsilon}}{\eta} + \frac{\eta}{2} T_B (c^2 + 1),$$

where $T_B$ is the number of points $x_t$ falling into ball $B$. Now, taking into account that $\tilde{L}$ is $\frac{2}{\gamma}$-Lipschitz, and that the functions $h$ and $r$ are assumed to be $L_\epsilon$-Lipschitz on $\mathcal{X}$, a direct adaptation of the proof of Theorem 1 in (Cesa-Bianchi et al., 2017) gives the bound

$$\sup_{\xi \in \mathcal{E}} \mathbb{E} \left[ \sum_{t=1}^{T} \tilde{L}(\xi_{t_\epsilon}, z_t) - \sum_{t=1}^{T} \tilde{L}(\xi, z_t) \right] \leq \frac{N_T \log K_{\epsilon}}{\eta} + \frac{\eta}{2} T(c^2 + 1) + L_\epsilon \frac{2}{\gamma} T,$$
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being \( N_T \leq C \varepsilon^{-d} \) the maximum number of balls created by \textsc{Contexp3-abs}. Using \( c \leq 1 \) and setting \( \eta = \sqrt{N_T \log K_\varepsilon} / T \) yields

\[
\sup_{\xi \in \mathcal{E}} \mathbb{E} \left[ \sum_{t=1}^{T} L(\xi_t, z_t) - \sum_{t=1}^{T} \tilde{L}(\xi_t, z_t) \right] \leq 2 \sqrt{T N_T \log K_\varepsilon} + L_\mathcal{E} \varepsilon + \frac{2}{\gamma} T.
\]

Next, optimizing for \( \varepsilon \) by setting \( \varepsilon \approx T^{-\frac{d+1}{2d+2}} \left( \frac{1}{\gamma} \right)^{\frac{d}{d+2}} \) (and disregarding \( L_\mathcal{E} \) and log factors) gives

\[
\sup_{\xi \in \mathcal{E}} \mathbb{E} \left[ \sum_{t=1}^{T} \tilde{L}(\xi_t, z_t) - \sum_{t=1}^{T} \tilde{L}(\xi_t, z_t) \right] = \tilde{O} \left( T^{\frac{d+1}{2d+2}} \left( \frac{1}{\gamma} \right)^{\frac{d}{d+2}} \right).
\]

Finally, we are left with connecting the above bound on the regret with a bound on the regret for \( L \). Now, observe that

\[
\mathbb{E} \left[ \sum_{t=1}^{T} \tilde{L}(\xi_t, z_t) \right] \geq \mathbb{E} \left[ \sum_{t=1}^{T} L(\xi_t, z_t) \right],
\]

since \( \tilde{L}(\xi, z_t) \) is an upper bound on \( L(\xi, z_t) \) for any \( \xi \) and \( z_t \). Moreover, if we assume for the sake of brevity that the minima are reached (the general case is straightforward to handle in a similar way), we can define

\[
\xi^* = (h^*, r^*) = \arg\min_{\xi \in \mathcal{E}} \sum_{t=1}^{T} L(\xi, z_t), \quad \bar{\xi}^* = \arg\min_{\xi \in \mathcal{E}} \sum_{t=1}^{T} \tilde{L}(\xi, z_t).
\]

We denote by \( M_T^*(\gamma) \) the number of \( x_t \) such that \( |r^*(x_t)| \leq \gamma \). Then, we can write

\[
\sum_{t=1}^{T} \tilde{L}(\bar{\xi}^*, z_t) \leq \sum_{t=1}^{T} \tilde{L}(\xi^*, z_t) \leq \sum_{t: |r^*(x_t)| > \gamma} \tilde{L}(\xi^*, z_t) + M_T^*(\gamma) \quad \text{(since \( \tilde{L} \leq 1 \))}
\]

\[
= \sum_{t: |r^*(x_t)| > \gamma} L(\xi^*, z_t) + M_T^*(\gamma) \quad \text{(using (3))}
\]

\[
\leq \sum_{t=1}^{T} L(\xi^*, z_t) + M_T^*(\gamma).
\]

Combining with (4) and (5) gives the following regret bound

\[
\sup_{\xi \in \mathcal{E}} \mathbb{E} \left[ \sum_{t=1}^{T} L(\xi_t, z_t) - \sum_{t=1}^{T} L(\xi, z_t) \right] \leq \tilde{O} \left( T^{\frac{d+1}{2d+2}} \left( \frac{1}{\gamma} \right)^{\frac{d}{d+2}} \right) + M_T^*(\gamma),
\]

thereby concluding the proof. \( \square \)

\textbf{Remark 1} The reader should observe that, since the algorithm is competing against an uncountably infinite set of experts, the standard regret guarantee of \( \sqrt{T} \) that one can achieve in the finite case cannot be obtained in general (see, e.g., the lower bound on regret of \( T^{(d-1)/d} \) by (Hazan & Megiddo, 2007), which holds in the easier full information setting). Notice that, while our algorithm \textsc{Contexp3-abs} admits a slightly worse bound of the form \( T^{(d+1)/(d+2)} \), it has the advantage of being computationally feasible. In particular, the covering of the input space \( \mathcal{X} \) can be done adaptively, as the points \( x_t \) are observed. In doing so, the number of \( \varepsilon \)-balls allocated can never exceed the total number of rounds \( T \). Given a new \( x_t \), the algorithm has to decide if a new ball needs to be created or an old ball can be used. Known data-structures exist to efficiently implement this decision (e.g., (Clarkson, 2006)). The extra additive term \( M_T^*(\gamma) \) in Theorem 2 is due to the
We now prove the theorem for UCBs where \( L \) is not Lipschitz. In fact, one can further improve the term \( T^{\frac{d+1}{d}} \) to \( T^{\frac{d}{d+1}} \) by adopting a hierarchical covering technique of the function space \( E \), each layer of the hierarchy being a pool of experts for the layer above it, see, e.g., (Cesa-Bianchi et al., 2017). However, the resulting algorithm would be of theoretical interest only, since it would be computationally very costly.

C. Additional material for the stochastic setting

In this section, we present the proofs of the theoretical guarantees for UCB-NT and UCB-GT, as well as the proof of Proposition 1. The following theorems hold more generally with \( S_j = \sqrt{\frac{\log t}{d_j}} \) for \( \beta > 2 \), which implies slightly better constants in the regret bound. However, for the sake of the simplicity of the presentation, below we set \( \beta = \frac{d}{2} \). Moreover, we prove Theorem 3 for the abstention loss \( L \), but it holds for any general loss function.

C.1. Regret of UCB-NT

We now prove the theorem for UCB-NT based on the admissible \( p \)-partitioning of the time-varying feedback graphs.

Proof of Theorem 3.

Proof. Consider a sequence of graph realizations \( G_1, \ldots, G_t \) denoted by \( G_t \). By conditioning on this quantity, the regret can be decomposed according to each arm \( i \):

\[
\sum_{t=1}^{T} E[L(\xi_t, z_t) - L(\xi_*, z_t)] = \sum_{t=1}^{T} E[E[L(\xi_t, z_t) - L(\xi_*, z_t) | G_t]]
\]

\[
= \sum_{t=1}^{T} E \left[ \sum_{i=1}^{K} E[L(\xi_i, z_t) - L(\xi_*, z_t) | G_t] E[1_{I_t = i} | G_t] \right]
\]

\[
= \sum_{t=1}^{T} \sum_{i=1}^{K} E[L(\xi_i, z_t) - L(\xi_*, z_t) | G_t] E[1_{I_t = i} | G_t] = E \left[ \sum_{i=1}^{K} \sum_{t=1}^{T} \Delta_i E[1_{I_t = i} | G_t] \right]
\]

where, in the last step, we used the fact that \( L(\cdot, z_t) \)s are independent of \( G_t \) since, by assumption, \( G_t \) only depends on information up \( t - 1 \). Next, we focus on bounding \( \sum_{t=1}^{T} E[1_{I_t = i} | G_t] \) for each arm \( i \).

We split the expectation according to the events \( Q_{i,t-1} > s_i \) and \( Q_{i,t-1} \leq s_i \), where \( s_i \) is a quantity determined later:

\[
\sum_{t=1}^{T} E[1_{I_t = i} | G_t] = \sum_{t=1}^{T} E[1_{I_t = i} (1_{Q_{i,t-1} \leq s_i} + 1_{Q_{i,t-1} > s_i}) | G_t]
\]

\[
\leq s_i + \sum_{t=1}^{T} E[1_{I_t = i} 1_{Q_{i,t-1} > s_i} | G_t].
\]

We wish to choose \( s_i \) sufficiently large so that the second term is bounded but so that it admits a mild dependence on \( T \). Now, whenever \( I_t = i \), by the design of the algorithm, it must be the case that the upper confidence bound of \( i \) is smaller than that of any other expert. Thus,

\[
E[1_{I_t = i} 1_{Q_{i,t-1} > s_i} | G_t] = P[I_t = i, Q_{i,t-1} > s_i | G_t] \leq P[\hat{\mu}_{i,t-1} - S_{i,t-1} \leq \hat{\mu}_* - S_{*,t-1}, Q_{i,t-1} > s_i | G_t],
\]

where \( * \) denotes the best-in-class expert. We now use the terms \( \mu_*, \mu_i \) and \( S_{*,t-1} \) to reorder the first event in the probability on the right-hand side of the last expression as follows:

\[
0 \leq \hat{\mu}_{*,t-1} - S_{*,t-1} - \hat{\mu}_* - S_{*,t-1} + S_{i,t-1}
\]

\[
\iff 0 \leq (\hat{\mu}_{*,t-1} - S_{*,t-1} - \mu_*) + (\mu_* - \hat{\mu}_* - S_{i,t-1} - 2S_{i,t-1}) + (\mu_* - \mu_i + 2S_{i,t-1})
\]

If we can show that the third term is negative, then the first and second term must be positive. Moreover, we will further show that the first and second terms can only be positive with an extremely low probability that is bounded by a constant.
We can bound these last probabilities using the union bound and a concentration inequality such as Hoeffding’s Inequality:

\[ \hat{\mu}_t = \frac{20 \log(T)}{\Delta_t^2}, \]

we ensure that the event \( Q_{i,t-1} > s_i \) implies that

\[ Q_{i,t-1} > \frac{20 \log(t)}{\Delta_t^2} \iff \mu_s - \mu_i + 2S_{i,t-1} < 0. \]

As explained above, it then follows that

\[
P[\hat{\mu}_t - S_{i,t-1} \leq \hat{\mu}_s - S_{s,t-1}, Q_{i,t-1} > s_i | G_t] \leq P[\hat{\mu}_s - S_{s,t-1} - \mu_s \geq 0 | G_t] + P[\hat{\mu}_t - \hat{\mu}_i - S_{i,t-1} - 2S_{i,t-1} \geq 0 | G_t].
\]

We can bound these last probabilities using the union bound and a concentration inequality such as Hoeffding’s Inequality:

\[
P[\hat{\mu}_t - S_{i,t-1} + S_{i,t-1} - 2S_{i,t-1} \geq 0 | G_t] = P\left[ -\frac{1}{Q_{i,t-1}} \sum_{s=1}^{t-1} L(\xi_t, z_s) 1_{i \in N_s(I_s)} + \mu_i - \sqrt{\frac{5 \log(t)}{Q_{i,t-1}}} \geq 0 | G_t \right].
\]

Now, the estimate \( \hat{\mu}_t, t-1 \) is an average of i.i.d. realizations of the random variable \( L(\xi_t, z) \), with \( z \sim D \), since the out-neighborhood of the chosen expert only depends on previous observations. That is,

\[
\frac{\mathbb{E}\left[ \sum_{s=1}^{t-1} L(\xi_t, z_s) 1_{i \in N_s(I_s)} \right]}{\mathbb{E}\left[ \sum_{s=1}^{t-1} 1_{i \in N_s(I_s)} \right]} = \frac{\mathbb{E}\left[ \sum_{s=1}^{t-1} L(\xi_t, z_s) 1_{i \in N_s(I_s)} | i \in N_s(I_s) \right]}{\mathbb{E}\left[ \sum_{s=1}^{t-1} 1_{i \in N_s(I_s)} | i \in N_s(I_s) \right]} = \frac{\mathbb{E}\left[ \sum_{s=1}^{t-1} 1_{i \in N_s(I_s)} L(\xi_t, z_s) | i \in N_s(I_s) \right]}{\mathbb{E}\left[ \sum_{s=1}^{t-1} 1_{i \in N_s(I_s)} \right]} = \mathbb{E}\left[ L(\xi_t, z) \right].
\]

Hence, \( \hat{\mu}_t, t-1 \) can be turned into an empirical estimate of \( \mu_i \) using the union bound as follows:

\[
P\left[ -\frac{1}{Q_{i,t-1}} \sum_{s=1}^{t-1} L(\xi_t, z_s) 1_{i \in N_s(I_s)} + \mu_i - \sqrt{\frac{5 \log(t)}{Q_{i,t-1}}} \geq 0 | G_t \right] \leq P\left[ \exists n \in [1, t] : -\hat{\mu}_t^n + \mu_i - \sqrt{\frac{5 \log(t)}{n}} \geq 0 | G_t \right]
\]

\[
\leq \sum_{n=1}^{t} \frac{1}{t^2} = \frac{1}{t^2},
\]

where \( \hat{\mu}_t^n = \frac{1}{n} \sum_{s=1}^{n} L(\xi_t, z_s) \). By the same reasoning, we can also bound the probability of the best arm:

\[
P\left[ \hat{\mu}_s, t-1 - S_{s,t-1} - \mu_s \geq 0 | G_t \right] \leq \sum_{n=1}^{t} \frac{1}{t^2} = \frac{1}{t^2}.
\]

By assumption, for each \( C_{t,k}, \forall i, j \in C_{t,k} \), it is the case that \( i \in N_s(j) \). Moreover, for any \( i \in [K] \), it must be the case that for every \( s \in [t], i \in C_{s,k} \) for some \( k \in [p] \). With these clusters \( C_{s,k} \), we can write

\[
Q_{i,t} = \sum_{s=1}^{t} 1_{i \in N_s(I_s)} = \sum_{s=1}^{t} \sum_{j=1}^{K} 1_{i \in N_s(j)} 1_{i \in C_{s,k}} = \sum_{s=1}^{t} \sum_{j=1}^{K} 1_{i \in N_s(j)} 1_{i \in C_{s, k}} = \sum_{s=1}^{t} \sum_{j=1}^{K} 1_{i \in N_s(j)} 1_{i \in C_{s, k}} = \sum_{s=1}^{t} \sum_{j=1}^{K} 1_{i \in N_s(j)} 1_{i \in C_{s, k}} = \sum_{s=1}^{t} \sum_{j=1}^{K} 1_{i \in N_s(j)} 1_{i \in C_{s, k}},
\]

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which implies that
\[
\sum_{i=1}^{K} \sum_{t=1}^{T} \Delta_i \mathbb{E}[1_{I_t=i} \mathbb{1}_{Q_{i,t-1} \leq s_i} | G_t] \leq \sum_{k \in [p]} \max_{i \in [T]} \Delta_j \sum_{t=1}^{T} \sum_{j \in \mathcal{E}_{t,k}} \mathbb{E} \left[ 1_{I_t=j} \mathbb{1}_{Q_{j,t-1} \leq \max_{j \in \mathcal{E}_{t,k}} s_j} | G_t \right] \leq \sum_{k \in [p]} \max_{i \in [T]} \Delta_j \left( \max_{j \in \mathcal{E}_{t,k}} s_j \right).
\]

Combining the above calculations, applying our definition for \( s_t \), and using the fact that the above analysis holds for any such partition shows that
\[
\mathbb{E} \left[ \min_{\{ \mathcal{E}_{t,k} \}} \sum_{p} \max_{i \in [T]} \Delta_j \left( \max_{j \in \mathcal{E}_{t,k}} s_j \right) \frac{20 \log(T)}{\Delta_j^2} + 5K \right].
\]

which implies the bound of the theorem. \( \square \)

**C.2. Regret of UCB-GT**

Next, we prove the regret bound for UCB-GT, which demonstrates how one can exploit the bias and feedback structure in the problem.

**Proof of Theorem 4.**

**Proof.** As in the previous proof, we focus on bounding \( \sum_{t=1}^{T} \mathbb{E}[1_{I_t=i} | G_t] \) for each arm \( i \). We again split the expectation according to the events based on \( Q_{i,t-1} \) as follows:
\[
\sum_{t=1}^{T} \mathbb{E}[1_{I_t=i} \mathbb{1}_{Q_{i,t-1} \leq s_i} | G_t] + \mathbb{E}[1_{I_t=i} \mathbb{1}_{Q_{i,t-1} > s_i} | G_t],
\]

where \( s_i \) is to be determined later. We then bound the second term using the algorithm’s choice of arm, \( I_t \):
\[
\mathbb{E}[1_{I_t=i} \mathbb{1}_{Q_{i,t-1} > s_i} | G_t] = \mathbb{P}[I_t = i, Q_{i,t-1} > s_i | G_t] \leq \mathbb{P}[\tilde{\mu}_{i,t-1} - S_{i,t-1} \leq \tilde{s}_i, Q_{i,t-1} > s_i | G_t],
\]

\( \tilde{\mu}_{i,t-1} \) is a biased estimate of \( \mu_i \). This is because whenever \( x_s \) falls in the region \( \{ x : r_i(x) > 0 \wedge r_{I_t}(x) \leq 0 \} \) and the condition \( \tilde{p}_{i,s}^{-1} \leq \gamma_{i,s-1} \) holds, the label \( y_s \) is not accessible. In this case, the UCB-GT algorithm updates the average loss of expert \( i \) optimistically, as if the expert were correct at that time step.

We can decompose this biased estimate \( \tilde{\mu}_{i,t-1} \) into two terms: \( \tilde{\mu}_{i,t-1} = \tilde{\mu}_{i,t-1} - \varepsilon_{i,t-1} \). The first term, \( \tilde{\mu}_{i,t-1} \), is an unbiased estimate of arm \( i \) and similar to the estimates in Theorem 3. The second term is the misclassification rate \( \varepsilon_{i,t-1} \) over \( \{ s \in [t-1] : r_i(x_s) > 0 \wedge r_{I_t}(x_s) \leq 0 \} \) whenever the condition \( \tilde{p}_{i,s}^{-1} \leq \gamma_{i,s-1} \) holds, that is, \( \varepsilon_{i,t-1} = \frac{1}{Q_{i,t-1}} \sum_{s=1}^{t-1} 1_{y_s h_t(x_s) \leq 1} 1_{r_i(x_s) > 0 \wedge r_{I_t}(x_s) \leq 0} \tilde{p}_{i,s}^{-1} \leq \gamma_{i,s-1} \).

Now, by the design of the UCB-GT, if arm \( i \) is chosen at time \( t \), it must be the case that \( \tilde{\mu}_{i,t-1} - S_{i,t-1} \leq \tilde{\mu}_{s,t-1} - S_{s,t-1} \). We can expand and rewrite this expression as follows:
\[
0 \leq \tilde{\mu}_{s,t-1} + \varepsilon_{i,t-1} - \varepsilon_{i,t-1} - S_{s,t-1} - \tilde{\mu}_{i,t-1} - \varepsilon_{i,t-1} + \varepsilon_{i,t-1} + S_{i,t-1} \\
\iff 0 \leq (\tilde{\mu}_{s,t-1} - S_{s,t-1} - \mu_s) + (\mu_i - \tilde{\mu}_{i,t-1} + S_{i,t-1} - 2S_{i,t-1}) + (\mu_s - \mu_i + (2 + C)S_{i,t-1}),
\]
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where we used the fact that \(-\varepsilon_{i,t-1} \leq 0\), and where we bounded \(\varepsilon_{i,t-1}\) as follows:

\[
\varepsilon_{i,t-1} = \frac{1}{Q_{i,t-1}} \sum_{s=1}^{t-1} 1_{y_s, h_s(x) \leq 0} 1_{r(x_s) > 0} 1_{r(x_s) \leq 0} \hat{P}_{i,s}^{-1} \leq \gamma_{i,s-1} = \frac{1}{Q_{i,t-1}} \sum_{s=1}^{t-1} \sum_{\xi_j \in \mathcal{E}} 1_{r(x_s) > 0} 1_{r(x_s) \leq 0} \hat{P}_{i,s}^{-1} \leq \gamma_{i,s-1} 1_{P=i}.
\]

The condition \(\hat{P}_{i,s}^{-1} \leq \gamma_{i,s-1}\) is equivalent to \(\sum_{s=1}^{t-1} 1_{r(x_s) > 0, r(x_s) \leq 0} \leq (s-1) \gamma_{i,s-1}\). Since the sum above is non-zero only when this condition holds, there exists \(s_j \in [1, t-1]\) such that \(\sum_{s=1}^{t-1} 1_{r(x_s) > 0, r(x_s) \leq 0} \hat{P}_{i,s}^{-1} \leq \gamma_{i,s} \leq (s_j - 1) \gamma_{i,s-1} + 1\). Moreover, using the fact that \((s_j - 1) \gamma_{i,s-1} = \sqrt{5Q_{i,s-1}} \log(s_j)/(K-1) \leq \sqrt{5Q_{i,s-1}} \log(t-1)/(K-1)\), we can conclude that

\[
\varepsilon_{i,t-1} \leq \frac{1}{Q_{i,t-1}} \sum_{\xi_j \in \mathcal{E}} \sum_{s=1}^{t-1} 1_{r(x_s) > 0, r(x_s) \leq 0} \hat{P}_{i,s}^{-1} \leq \frac{K-1}{Q_{i,t-1}} \sqrt{5Q_{i,s-1} \log(t-1)/(K-1)} + 1 \leq C \sqrt{\frac{5\log(t-1)}{Q_{i,t-1}}}
\]

for some constant \(C > 0\). The rest of the proof now follows by similar arguments as in the proof of Theorem 3. Specifically, we can choose \(s_j\) such that the term \(\mu_{s_j} - \mu_i + (2 + C)\xi_{i,s-1}\) is negative, and since now \(\hat{P}_{i,s-1}^{-1}\) and \(\mu_{i,t-1}\) are unbiased estimates, we can bound the probabilities \(\mathbb{P}[\hat{P}_{i,t-1} - S_{i,t-1} - \mu_s \geq 0|G_t]\) and \(\mathbb{P}[\mu_i - \hat{P}_{i,t-1} - S_{i,t-1} \geq 0|G_t]\) using standard concentration inequalities. \(\square\)

C.3. Linear regret without the subset property

In this section, we prove Proposition 1, which shows that when the subset property does not hold for a feedback graph, then it is possible to incur linear regret.

**Proof of Proposition 1.**

*Proof. Let \(p^* \in (0, 1)\). We design a setting in which with probability at least \(p^*\), the UCB-NT algorithm incurs linear regret.

Since the family of abstention functions induces a feedback graph that violates the subset property, there exist pairs \((h_i, r_i)\) and \((h_j, r_j)\) and points \(x^*, \bar{x}\) for which \(x^* \in \mathcal{A}_i \setminus \mathcal{A}_j, \bar{x} \in \mathcal{A}_i \cap \mathcal{A}_j\), where \(\mathcal{A}_i\) and \(\mathcal{A}_j\) are the acceptance regions associated with \(r_i\) and \(r_j\), respectively, and the feedback graph is designed such that the algorithm updates the pair \((h_i, r_i)\) when the pair \((h_j, r_j)\) is selected.

Now, for some \(p \in (0, 1)\) to be determined later, consider a distribution with probability \(p\) on \((\bar{x}, \bar{y})\) and \((1 - p)\) on \((x^*, y^*)\).

We choose the set of hypothesis functions \(\mathcal{H} = \{h_i, h_j\}\), the loss function \(\ell\) in (1), and the labels \(y^*\) and \(\bar{y}\) in such a way that \(\ell(h_i, \hat{y}(\bar{x})) = c - \beta, \ell(h_j, \bar{y}(\bar{x})) = c - \alpha,\) and \(\ell(y^*, h_j(x^*)) = 0\), where \(\alpha, \beta\) are values that will be later specified. For instance, we can consider the hinge loss \(\ell(y, \bar{y}) = (1 - y \bar{y})_+\), and \(h_i, h_j\) such that \(h_i(\bar{x}) = \frac{1-c+\beta}{y}, h_j(\bar{x}) = \frac{1-c+\alpha}{y}\), and \(h_i(x^*) = \frac{1}{y}\). Note that, since \(r_j(x^*) < 0, \ell(y^*, h_j(x^*))\) may admit any value.

Now, by construction, \(\mu_i = (c - \beta)p\) and \(\mu_j = (c - \alpha)p + c(1-p) = c - \alpha p\). We claim that we can choose \(\alpha, \beta, \) and \(p\) such that (1) \(\alpha > \beta\); (2) \(\mu_i < \mu_j\); (3) \(\mu_j < \ell(\bar{y}, \hat{h}_i(\bar{x}))\)

The first condition is immediate. The second condition is equivalent to \(c - \beta p < c - \alpha p\), which is itself equivalent to \(\alpha - \beta < \frac{c(1-p)}{p}\). By continuity, we can choose \(\alpha\) and \(\beta\) close enough such that this is true for any \(p \in (0, 1)\). The third condition is equivalent to \(c - \alpha p < c - \beta\), which is itself equivalent to \(\beta < \alpha p\). This is true for \(p\) close enough to 1.

Now let \(n \in \mathbb{N}\) large enough such that \(\mu_j < \ell(\bar{y}, \hat{h}_i(\bar{x})) - \sqrt{\frac{5\log(n)}{n}}\). By continuity, we can choose \(p\) large enough such that \(p > (p^*)^{1/n}\), and for this choice of \(p\), we can choose \(\alpha\) and \(\beta\) such that \(\alpha > \beta, \alpha, \beta < c, \alpha - \beta < \frac{c(1-p)}{p}, \) and \(\beta < \alpha p\).

For instance, if we, without loss of generality, assume that \(p > \frac{1}{2}\), then we can choose, \(\alpha = \frac{c(1-p)}{2p}\) and \(\beta = \frac{c(1-p)}{4}\).

Then, with probability \(p^n > p^*,\) the point \(\bar{x}\) will be sampled \(n\) times at the start of the game, such that the pair \((h_j, r_j)\) will
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<table>
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<tr>
<th>Dataset</th>
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<tr>
<td>ijcnn</td>
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<tr>
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<tr>
<td>eye</td>
<td>14</td>
</tr>
<tr>
<td>CIFAR</td>
<td>25</td>
</tr>
</tbody>
</table>

Table 1: Table shows the number of features of each dataset.

have a lower confidence bound than the pair $(h_i, r_i)$ at all time steps. Thus, UCB-NT will choose the pair $(h_j, r_j)$ throughout the entire game, even though $\mu_i < \mu_j$. Consequently, the regret of the algorithm will be at least $T(\mu_j - \mu_i)$. □

D. Additional experimental results

In this section, we present several figures showing our experimental results. Figure 7 and Figure 8 show the regret for different abstention costs $c \in \{0.1, 0.2, 0.3\}$ for all our datasets. We observe that, in general, UCB-GT outperforms UCB-NT and UCB for all datasets and is even within the standard deviation of the FS's regret for some of datasets. The figures also indicate that the regret of UCB decreases slowly. This is expected, since there are 2,100 experts, 10,000 time steps, and the algorithm only updates a single expert per time step.

Figure 9 and Figure 10 show the fraction of abstained points for all the datasets. Figure 11 also shows how the fraction of abstained points varies with abstention cost for two extreme values $c \in \{0.001, 0.9\}$. Again UCB-GT admits a lower regret than UCB-NT and UCB and, as expected, the fraction of points decreases as the cost of abstention increases. Figure 12 shows the effect of using confidence-based experts and suggests that the choice of experts does not affect the relative performance of the algorithms. We also tested the effect of varying the number of experts: Figure 13 shows the regret of three datasets when the number of experts is $K = 500$ and $T = 5,000$. For this set of experts, we find a similar pattern of performance as above.

Next, we describe in more detail the datasets and how they were processed. In Table 1, we show the number of features for each dataset. For all datasets, we normalized the features to be in the interval $[-1, 1]$. Note that the reason for choosing abstention functions with radius range $(0, \sqrt{d})$ is to cover the entire hypercube $[-1, 1]^d$ with our concentric annuli. For the CIFAR dataset, we extracted the first twenty-five principal components of the horse and boat images, projected the images on these components, and normalized the range of the projections to $[-1, 1]$. The features of the synthetic dataset are drawn from the uniform distribution over $[-1, 1]^2$ and the label is determined by the sign of the projection of a point onto the normal of the diagonal hyperplane $y = -x$.

The confidence-based abstention function has the form $r(x) = |h(x)| - \theta$. In our experiments (Figure 12), we generated twenty abstention functions with thresholds $\theta \in (0, \ldots, 0.25)$, which are paired with each predictor. The predictors are axis-aligned planes along each feature of the dataset. For each dataset, the number of predictors is $\lceil 100/d \rceil$ where $d$ is the dimension of the dataset. We chose twenty abstention functions and about 100 prediction functions in order to match the experimental setup of the randomly drawn experts. The total number of experts is then $\lceil 100/d \rceil \cdot 20 \cdot d$. Note that we only tested some of our datasets since for larger dimensions $d$, the number of experts per feature was too small.
D.1. Average regret for different abstention costs and datasets

Figure 7: A graph of the averaged regret \( R_t(\cdot)/t \) with standard deviations as a function of \( t \) (log scale) for UCB-GT, UCB-NT, UCB, and FS for different values of abstention costs. Each row is a dataset, starting from the top row we have: CIFAR, ijcn, HIGGS, phishing, and covtype.
Figure 8: A graph of the averaged regret $R_t(\cdot)/t$ with standard deviations as a function of $t$ (log scale) for UCB-GT, UCB-NT, UCB, and FS for different values of abstention costs. Each row is a dataset, starting from the top row we have: eye, cod-ran, synthetic, skin, and guide.
D.2. Average fraction of abstention points for different abstention costs and datasets

Figure 9: A graph of the averaged fraction of abstained points with standard deviations as a function of $t$ (log scale) for UCB-GT, UCB-NT, UCB, and FS for different values of abstention costs. Each row is a dataset, starting from the top row we have: CIFAR, ijcnn, HIGGS, phishing, and covtype.
Figure 10: A graph of the averaged fraction of abstained points with standard deviations as a function of $t$ (log scale) for UCB-GT, UCB-NT, UCB, and FS for different values of abstention costs. Each row is a dataset, starting from the top row we have: eye, cod-ran, synthetic, skin, and guide.
D.3. Average regret and fraction of abstention points for extreme abstention costs

![Graphs showing regret and fraction abstained](image)

Figure 11: A graph of the averaged regret $R_t(\cdot)/t$ and fraction of points rejected with standard deviations as a function of $t$ (log scale) for UCB-GT, UCB-NT, UCB, and FS for different values of abstention costs. The fraction of points decreases as the cost of abstention increases. The UCB-GT outperforms UCB-NT and UCB while approaching the performance of FS even at these extreme values of $c$.

Each row is a dataset, starting from the top row we have: CIFAR, ijcnn, phishing, and covtype.
D.4. Average regret for confidence-based experts

Figure 12: A graph of the averaged regret $R_t(\cdot)/t$ with standard deviations as a function of $t$ (log scale) when using the confidence based experts for UCB-GT, UCB-NT, UCB, and FS. Each row is a dataset, starting from the top row we have: synthetic, skin, guide, ijcnn and CIFAR.
D.5. Average regret for a smaller set of experts

Figure 13: A graph of the averaged regret $R_t(\cdot)/t$ of abstained points with standard deviations as a function of $t$ (log scale) for UCB-GT, UCB-NT, UCB, and FS for different values of abstention costs. Each row is a dataset, starting from the top row we have: guide, synthetic, and skin. We used $K = 500$ experts and $T = 5,000$ rounds in order to see the effect when changing the number of experts used.
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