Mehryar Mohri
Advanced Machine Learning 2025
Courant Institute of Mathematical Sciences
Homework assignment 2
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Due: April 22, 2025

A Weighted online-to-batch

Let ℓ be a loss function convex with respect to its first argument and bounded by one. Let $h_1, \ldots, h_T$ be the hypotheses returned by an on-line learning algorithm $\mathcal{A}$ with regret R_T when sequentially processing $(x_t, y_t)_{t=1}^T$, drawn i.i.d. according to some distribution $\mathcal{D}$.

1. Fix some arbitrary non-negative weights $q_1, \ldots, q_T$ summing to one. Then, show that with probability at least $1 - \delta$, the hypothesis $h = \sum_{t=1}^T q_t h_t$ satisfies each of the following inequalities:

$$\mathop{\mathbb{E}}_{(x,y)\sim\mathcal{D}}[\ell(h(x), y)] \leq \sum_{t=1}^T q_t \ell(h_t(x_t), y_t) + \|q\|_2 \sqrt{2\log(1/\delta)}$$

$$\begin{aligned}\mathop{\mathbb{E}}_{(x,y)\sim\mathcal{D}}[\ell(h(x), y)] &\leq \inf_{h\in\mathcal{H}} \mathop{\mathbb{E}}_{(x,y)\sim\mathcal{D}}[\ell(h(x), y)] + \frac{R_T}{T}\\ &\quad + \|q - u\|_1 + 2\|q\|_2 \sqrt{2\log(1/\delta)},\end{aligned}$$

where q is the vector with components q_t and u the uniform vector with all components equal to $1/T$.

2. Here, we seek to prove a bound that holds uniformly for all weight vectors q in some set. To do so, we consider a weight vector p that serves as a *reference*. A natural reference in this context could be for example the uniform distribution. Show that, for any $\delta > 0$, the following holds with probability at least $1 - \delta$ for all $q \in \{q \colon \|q - p\|_1 < 1\}$:

$$\begin{aligned}\mathop{\mathbb{E}}_{(x,y)\sim\mathcal{D}}[\ell(h(x), y)] &\leq \sum_{t=1}^T q_t \ell(h_t(x_t), y_t) + 2\|q - p\|_1\\ &\quad + (\|q\|_2 + 2\|q - p\|_1)\left[2\sqrt{\log\log_2 \frac{2}{1 - \|q - p\|_1}} + \sqrt{2\log\frac{2}{\delta}}\right].\end{aligned}$$

Hint: consider the first inequality proven above for a fixed weight vector q^k and approximation error ϵ_k, for any $k \geq 0$. Show that the inequality can be extended to hold uniformly for all $k \geq 0$ if you choose $\epsilon_k = \epsilon + \sqrt{2\log(k+1)}$.

B Swap regret for large expert spaces

Leverage the results presented in class to give a swap regret algorithm tailored for large expert spaces. You should give a full description of your algorithm and provide a detailed proof of the corresponding regret guarantee.