Algorithmic Finance

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Contents

Chapter 1

Scribe: Ron Even, March 23, 1995

1.1 Preliminaries

For starters, we will examine a finite world, W, with S states, and N securities. Securities can be T-bills, commodities, precious metals, and currencies, as well as corporate stock. A portfolio is a collection of securities such as an index fund or a mutual fund. The states of the system are simply defined by the behavior of the securities. For example, a simple system may just have two states—"up" and "down." The state is "up" when the market goes up (Bullish) and down when the market is down (Bearish).

We have a *price vector*,

$$\vec{q} = (q_1, q_2, \ldots, q_N) \in \mathbb{R}^N,$$

which gives the current price of all N securities.

$$\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_N) \in \mathbb{R}^N \setminus \vec{0},$$

is a *portfolio* where θ_i is the number of units of security i in the portfolio. $\vec{q} \cdot \vec{\theta}$ gives the market price of a portfolio. Often, we examine θ such that $\vec{q} \cdot \vec{\theta} = 0$. By combining shorts (borrowing) and longs (lending) we can generate portfolios with a net cost of 0. Finally, we have a $N \times S$ *dividend matrix*,

$$D = \begin{bmatrix} D_{11} & D_{12} & \ldots & D_{1S} \\ D_{21} & D_{22} & \ldots & D_{2S} \\ \vdots & \vdots & \vdots & \vdots \\ D_{N1} & D_{N2} & \ldots & D_{NS} \end{bmatrix} \in \mathbb{R}^{N \times S}.$$

If we call the i^{th} column vector $\vec{D}_i$, then $\vec{D}_i \cdot \vec{\theta}$ is the value of the portfolio in state i.

1.2 Arbitrage

By an *arbitrage*, we mean one of two scenarios:

$$\left(\exists \vec{\theta}\right) \left[\vec{q} \cdot \vec{\theta} = 0 \quad \wedge \quad D^T\theta \in \mathbb{R}^S_{\geq 0} \setminus \{\vec{0}\}\right] \tag{1.1}$$

$$\left(\exists \vec{\theta}\right) \left[\vec{q} \cdot \vec{\theta} < 0 \quad \wedge \quad D^T\theta \in \mathbb{R}^S_{\geq 0}\right] \tag{1.2}$$

In the first scenario, we obtain our portfolio for a net cost of zero, yet we at least break even in each possible state and make a profit in at least one possible state. In the latter scenario, we have already made a profit as our portfolio has negative cost, and we are guaranteed to at least break even in each possible state of the world.

A convenient way of stating the no arbitrage condition is by saying that no matter what nontrivial portfolio is chosen (i.e., $\vec{\theta} \in \mathbb{R}^N \setminus \{\vec{0}\}$),

$$(-\vec{q} \cdot \vec{\theta}, D_1 \cdot \vec{\theta}, \ldots, D_S \cdot \vec{\theta}) \not\in \mathbb{R}^{S+1}_{\geq 0} \setminus \{\vec{0}\}.$$

In other words,

$$\left(\forall \vec{\theta} \in \mathbb{R}^N\right) \left[D^T\theta \in \mathbb{R}^S_{\geq 0} \to [\vec{q} \cdot \vec{\theta} > 0 \vee (D^T\theta = 0 \wedge \vec{q} \cdot \vec{\theta} = 0)]\right].$$

If we write H^+_θ for the halfspace $\{\vec{p} : \vec{p} \cdot \vec{\theta} \geq 0\}$ and its boundary as the hyperplane $\partial(H^+_\theta) = h_\theta$, then above statement has precisely the following meaning: For every $\vec{\theta}$, if

$$D_1, D_2, \ldots, D_S \in H^+_\theta,$$

then $\vec{q} \in \text{Interior} H^+_\theta$, provided that D_1, ..., D_S and $\vec{q}$ do not all lie on the hyperplane h_θ (a degenerate situation). Equivalently $\vec{q}$ is in the relative interior of

$$\bigcap_\theta \{H^+_\theta : D_1, D_2, \ldots, D_S \in H^+_\theta\},$$

the positive hull (or the cone generated by) D_1, D_2, ..., D_S. No arbitrage condition is equivalent to the geometric conditions:

$$\vec{q} \in \text{Interior PositiveHull}(D_1, \ldots, D_S),$$

or

$$\vec{0} \in \text{Interior ConvexHull}(-\vec{q}, D_1, \ldots, D_S).$$

The convex hull of a set of vectors is the smallest convex subset of $\mathbb{R}^N$ containing these vectors. The positive hull of a set of vectors is the smallest cone (with apex at origin) of $\mathbb{R}^N$ containing these vectors. Both sets are convex and closed.

A more intuitive proof of these facts in the nondegenerate situation is given in the next section.

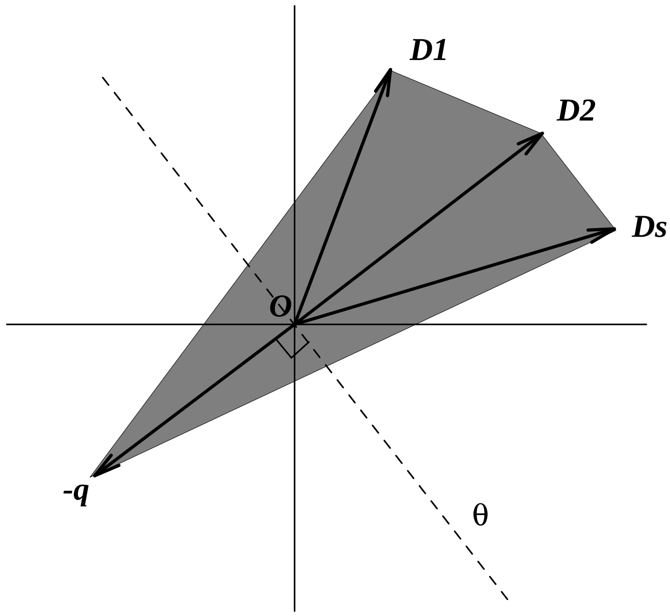


Figure 1.1: Diagram of no arbitrage situation for two assets ($N = 2$). The dotted line represents all the portfolios of net cost zero (i.e., $\vec{q} \cdot \vec{\theta} = 0$). The shaded region is the ConvexHull($-q$, D_1, …, D_S) and in this case, contains the origin in it interior.

1.3 No Arbitrage

For simplicity, we assume $S \geq N$ and that the vectors span (linearly) the entire space. Thus no arbitrage condition can be written as saying that for no nontrivial portfolio $\vec{\theta}$,

$$(-\vec{q} \cdot \vec{\theta} \geq 0) \ \wedge \ (D_1 \cdot \vec{\theta} \geq 0) \ \wedge \ \cdots \ \wedge \ (D_S \cdot \vec{\theta} \geq 0),$$

as it is guaranteed that for every portfolio one of the inequalities is strict.

Lemma 1.3.1 *There is no arbitrage iff*

$$\vec{0} \in \textit{Interior ConvexHull}(-\vec{q}, D_1, \ldots, D_S).$$

PROOF.
$\Longleftarrow$: Assume that there is an arbitrage. That is, there is a portfolio $\vec{\theta}$ satisfying all of the following conditions:

$$(-\vec{q} \cdot \vec{\theta} \geq 0) \ \wedge \ (D_1 \cdot \vec{\theta} \geq 0) \ \wedge \ \cdots \ \wedge \ (D_S \cdot \vec{\theta} \geq 0).$$

In particular if $\vec{p} \in$ ConvexHull($-\vec{q}$, D_1, …, D_S) [i.e., $\vec{p} = -\alpha_0 \vec{q} + \alpha_1 D_1 + \cdots + \alpha_S D_S$, with $\sum \alpha_i = 1$ and $\alpha_i \geq 0$] then $\vec{p} \cdot \theta \geq 0$. In other words, every point in the convex hull is in the halfspace H_θ^+ of $\mathbb{R}^N$ defined by $\{\vec{p} : \vec{p} \cdot \vec{\theta} \geq 0\}$. Thus it refutes the assumption that

$$\vec{0} \in \text{Interior ConvexHull}(-\vec{q}, D_1, \ldots, D_S).$$

$\Longrightarrow$: For no arbitrage case, we consider an arbitrary portfolio $\vec{\theta}$ and its negative $-\vec{\theta}$. Let us divide the vectors $\{-\vec{q}, D_1, \ldots, D_S\}$ into two subsets S_1 and S_2 (not necessarily, $S_1 \cap S_2 = \emptyset$) such that for every $\vec{p_1} \in S_1$, $\vec{p_1} \cdot \vec{\theta} \geq 0$ and for every $\vec{p_2} \in S_2$, $\vec{p_2} \cdot \vec{\theta} \leq 0$. By assumption, neither S_1 nor S_2 is empty, as this would imply either $\vec{\theta}$ or $-\vec{\theta}$ would result in an arbitrage situation. Thus, as a consequence of the no arbitrage condition, the hyperplane $h_\theta = \{\vec{p} : \vec{p} \cdot \vec{\theta} = 0\}$ would strictly separate the ConvexHull(S_1) and ConvexHull(S_2) and thus intersect ConvexHull$(S_1 \cup S_2)$. As this holds for any arbitrary portfolio, we get the desired consequence:

$$\vec{0} \in \text{Interior ConvexHull}(-\vec{q}, D_1, \ldots, D_S).$$

This proves the lemma. □

1.4 The State Price Vector

We derive what is known as the *state price vector* $\vec{\psi}$.

No arbitrage

$$\begin{aligned} &\Leftrightarrow \quad 0 \in \text{Interior ConvexHull}(-\vec{q}, D_1, \ldots, D_S) \\ &\Leftrightarrow \quad \left(\exists \vec{\alpha} \in \mathbb{R}^S_{>0}, \sum_{i=0}^{S} \alpha_i = 1\right) \left[-\alpha_0 \vec{q} + \alpha_1 D_1 + \cdots + \alpha_S D_S = 0\right] \end{aligned}$$

Thus,

$$\begin{aligned} \vec{q} &= \frac{\alpha_1}{\alpha_0} D_1 + \cdots + \frac{\alpha_S}{\alpha_0} D_S = \psi_1 D_1 + \cdots + \psi_S D_S \\ &= \frac{1}{R}(p_1 D_1 + \ldots + p_S D_S), \quad \text{where } \sum_{i=1}^{S} p_i = 1, \forall i, 0 < p_i \leq 1 \end{aligned}$$

In the above formula, R is interpreted as the discount on riskless borrowing, $\vec{\psi}$ is called the state price vector, and the p_i's are called *risk-neutral probabilities*. If $R = 1 + r$ then r is the interest rate.

Finally, we give an illustration of why r is the interest rate. For this, we examine a riskless portfolio, $\hat{\theta}$ which returns \$1 no matter what state we end up in. Its cost is $\vec{q} \cdot \hat{\theta}$.

$$\vec{q} \cdot \hat{\theta} = \sum_{i=1}^{N} \sum_{j=1}^{S} \frac{p_j}{R} D_{ij} \hat{\theta}_i$$

$$
\begin{aligned}
&= \frac{1}{R}\sum_{j=1}^{S} p_j \sum_{i=1}^{N} \hat{\theta}_i D_{ij} \\
&= \frac{1}{R}
\end{aligned}
$$

We see that the price of the portfolio is its expected future value (under risk-neutral probabilities) discounted by $R = 1 + r$, with $r =$ the interest rate.

A *state price vector* $\vec{\psi}$ is a vector in $\mathbb{R}^S_{>0}$ such that $\vec{q} = D\vec{\psi}$.

Corollary 1.4.1 *There is no arbitrage iff there is a state price vector.*
PROOF.
One direction is already shown in the preceding paragraph. Note that existence of state price vector implies that

$$\frac{1}{\sum \psi_i}(-\vec{q} + \psi_1 D_1 + \cdot\psi_S D_S) = 0$$

and $0 \in$ Interior ConvexHull$(-\vec{q},\ D_1,\ \ldots,\ D_S)$, a condition equivalent to "no-arbitrage condition." ☐

Chapter 2

Scribe: Toto Paxia, March 30, 1995

2.1 Weak Arbitrage

A dividend-price pair (D, q) is defined to be *weakly arbitrage-free* if

$$\left(\forall \vec{\theta} \in R^N\right) \left[D^T \vec{\theta} \in R^S_{\geq 0} \rightarrow \vec{q} \cdot \vec{\theta} \geq 0\right].$$

If H^+_θ is the halfspace $\{\vec{p} : \vec{p} \cdot \vec{\theta} \geq 0\}$, the above condition is equivalent to

$$\left(\forall \vec{\theta} \in R^N\right) \left[D_1, D_2, \ldots, D_S \in H^+_\theta \Leftrightarrow \vec{q} \in H^+_\theta\right].$$

Hence

$$\vec{q} \in \bigcap_\theta \{H^+_\theta : D_1, D_2, \ldots, D_S \in H^+_\theta\},$$

the positive hull (or the cone generated by) D_1, D_2, …, D_S. Weak-arbitrage-free condition is thus equivalent to the geometric conditions:

$$\vec{q} \in \text{PositiveHull}(D_1, \ldots, D_S),$$

Under this condition, the state price vector $\vec{\psi}$ can again be derived, but now in

$$\vec{q} = \psi_1 D_1 + \cdots + \psi_S D_S,$$

where the ψ_i's are ≥ 0. The vector $\vec{\psi} \in \mathbb{R}^S_{\geq 0}$ is called the *weak state price vector*.

2.2 Risk Neutral Portfolio

Any portfolio

$$\vec{\theta^0} = (\theta^0_1, \theta^0_2, \ldots, \theta^0_N),$$

is a *risk neutral portfolio* if in all the possible states $\in \{1, 2, \ldots, S\}$, one gets the same dividend:

$$D^T\theta^0 = (R^0, R^0, \ldots, R^0).$$

We can express the price paid for the risk neutral portfolio in terms of the state price vector, and the dividend:

$$\begin{aligned}
\vec{q} &= \psi_1 D_1 + \cdots + \psi_S D_S \\
\vec{q}\cdot\vec{\theta^0} &= \psi_1 D_1 \vec{\theta^0} + \cdots + \psi_S D_S \vec{\theta^0} \\
&= \psi_1 R^0 + \cdots + \psi_S R^0 = (\psi_1 + \cdots + \psi_S) R^0 = \frac{R^0}{R}
\end{aligned}$$

where $1/R = \sum \psi_i$. Thus we pay

$$\vec{q}\cdot\vec{\theta^0} = \frac{R^0}{R},$$

and get R^0 independent of the state of the world (risk neutral).

The above formula is the market value of the risk neutral portfolio. If we suppose that this value is unitary, then for any possible state

$$R^0 = R = \frac{1}{\psi_1 + \cdots \psi_N},$$

and the price can be written as

$$\begin{aligned}
\vec{q} &= \psi_1 D_1 + \cdots + \psi_S D_S \\
&= \frac{1}{R}(p_1 D_1 + \cdots + p_S D_S)
\end{aligned}$$

where the risk neutral probabilities are now

$$p_i = \frac{\psi_i}{\psi_1 + \cdots + \psi_s} = \psi_i R.$$

The discount factor R can also be written as

$$R = 1 + r,$$

where r is the "interest rate". The risk neutral probabilities defined above, are such that $p_i \in [0, 1[$ and $\sum_{i=1}^S p_i = 1$. If we consider the expected value relative to these "probabilities" p_i's, (the so-called *risk-neutral probabilities*) the price can be expressed as

$$\vec{q} = \frac{1}{R} E^P(D_i).$$

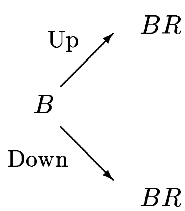
Thus, in the case of weak arbitrage, some of the risk neutral probabilities could be equal to zero.

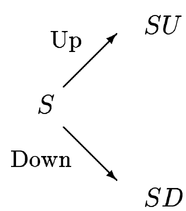
2.3 Examples

In the following examples we will consider a market with two kind of securities:

- A risk-free asset B (Bond).
- A risky asset S (Stock).

The market has only two possible states: Up and Down. Let also R be the discount factor. The diagrams

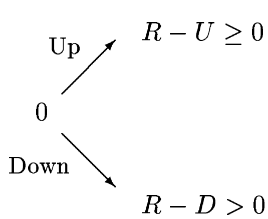




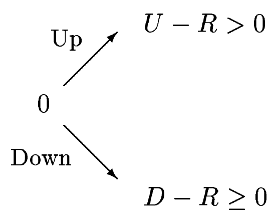
show that in the case of the Bond, in all the possible states of the market, the payoff will be BR. In the stock case, the payoff will depend on the future state of the market. We will assume that

$$D < R < U,$$

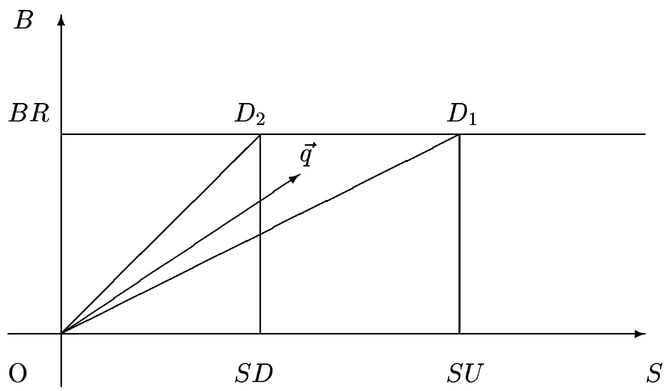
otherwise if we suppose for example $R \geq U > D$, then we could create a portfolio $\theta = (\frac{1}{B}, -\frac{1}{S})$ described in the following diagram:



which is an arbitrage. similarly, if $U > D \geq R$, than the arbitrage will be obtained with the portfolio $\theta = (-\frac{1}{B}, \frac{1}{S})$



So we must have $D < R < U$. A geometrical representation is



where the price vector must lie inside the cone OD_1D_2. Then

$$q = \psi_1 D_1 + \psi_2 D_2$$

$$\begin{pmatrix} B \\ S \end{pmatrix} = \psi_1 \begin{pmatrix} BR \\ SU \end{pmatrix} + \psi_2 \begin{pmatrix} BR \\ SD \end{pmatrix}$$

The solution of this linear system is given by

$$\psi_1 = \frac{1}{R} \frac{R - D}{U - D}$$

$$\psi_2 = \frac{1}{R} \frac{U - R}{U - D}.$$

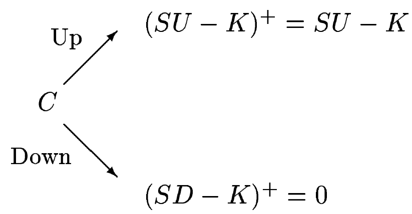
Now the discount factor R, can be viewed as the *barycenter* between U and D with "weights" ψ_1 and ψ_2. Now, it is easy to compute the risk neutral probabilities, since

$$q = \frac{1}{R}(p_1 D_1 + p_2 D_2),$$

$$\text{with} \quad p_1 = \frac{R - D}{U - D} \quad p_2 = \frac{U - R}{U - D}.$$

2.3.1 Call Option

The next example is a *call option* on a stock S, with *strike price* $K > SD$.

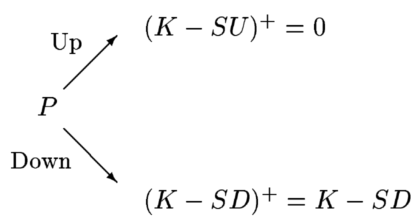


Here the price paid for the portfolio is C. If the future state is Up, then the payoff will be $SU - K$, otherwise it will be 0 (note that $SD - K$ is ≤ 0). The price in this case can be expressed using the risk neutral probabilities as

$$C = \frac{1}{R} \frac{R - D}{U - D} (SU - K).$$

2.3.2 Put Option

The converse *put option* is represented as



with

$$P = \frac{1}{R} \frac{U - R}{U - D} (K - SD).$$

2.3.3 Put-Call Parity

Suppose that we buy a call and sell a put option. Then the net price paid will be

$$\begin{aligned} C - P &= \frac{p_1}{R}(SU - K) - \frac{p_2}{R}(K - SD) \\ &= \frac{p_1 \cdot SU + p_2 \cdot SD}{R} - \frac{p_1 + p_2}{R} K \\ &= S - \frac{K}{R} \end{aligned}$$

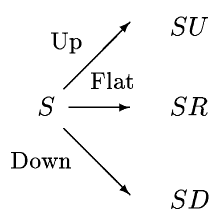
We just proved that buying a call and selling a put, is equivalent to buy a stock and sell a discounted bond. Also

$$C = S + P - \frac{K}{R}$$

can be used to express the price of a call. This formula is referred as "put-call parity."

2.3.4 Volatility and Call-Put Straddle

We can now change our model for the market. In particular we can augment the number of possible states. Now the states set S has three elements: Up, Down and Flat:

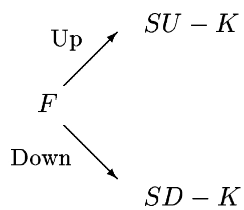


Suppose that the probabilities of the three states are (p_u, p_f, p_d). Suppose they take two possible sets of values: $(1/3, 1/3, 1/3)$ and $(1/2, 0, 1/2)$. In either case, we see that $R = (U + D)/2$. Now if we have a call and a put option on this with strike price $K = SR = S(U + D)/2$, then the call and put under the first probabilities are priced as: $C = (SU - K)/3R$ and $P = (K - SD)/3R$. Thus $C + P = S(U - D)/3R$. However, if the second probabilities are the real probabilities (i.e., the stock is more *volatile* than normally assumed) then the holder of the portfolio with one call and one put will get $\min(SU - K, K - SD) = S(U - D)/2$, independent of whether market goes up or down. This pay off is no worse than any risk neutral portfolio of cost $S(U - D)/2R$.

Thus, if we believe that the risk neutral probabilities are $(1/2, 0, 1/2)$, then we can do a *call-put straddle*. If the risk neutral probabilities are really $(1/2, 0, 1/2)$, this creates an arbitrage situation.

2.3.5 Forward Contract

Another example is the *forward contract*:



with

$$\begin{aligned} F &= \frac{p_1}{R}(SU - K) \;+\; \frac{p_2}{R}(SD - K) \\ &= S - \frac{K}{R} \end{aligned}$$

2.3.6 St. Petersburg's Paradox

Suppose now that we have a *lottery*, where we bet a fixed amount of money (to be determined). Then we toss a coin. If we get the head we will be paid 2. If we get a tail and then a head, we will be paid 4. In general the sequence

$$T^{(1)} \cdots T^{(i)} H$$

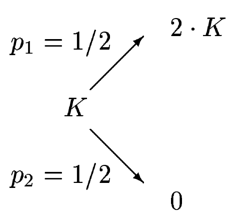
will be paid 2^{i+1}, and has probability $\frac{1}{2^{i+1}}$. If we compute the expected value we get infinity:

$$\sum_{i=1}^{\infty} \frac{1}{2^{i+1}} 2^{i+1} = \infty.$$

Thus this asset will cost us infinite amount of money, while only paying some finite amount of dividend. This is usually referred to as St. Petersburg's paradox.

2.3.7 Martingale Game

next consider the *Martingale game*, which is represented in the following diagram:



Suppose our strategy is to repeatedly play this game each time doubling the amount of money we put in (i.e, 1 \$, 2 \$, 4 \$, ...), until we "win" (i.e., take the up-arrow in the diagram and get a pay-off of double the amount we put in) and then stop. Assuming that we win at the nth trial, we would have paid

$$1 + 2 + 4 + \cdots + 2^{n-1} = (2^n - 1)\$$$

but get $2 \cdot 2^{n-1} = 2^n$ \$. Thus we are guaranteed to have a profit of 1 \$ eventually.

Chapter 3

Scribe: Amy Greenwald, April 6, 1995

3.1 Real Probabilities

Recall from Lecture 2 that today's price is the expectation of tomorrow's dividend divided by the discount factor:

$$\vec{q} = \frac{1}{R} E^P(D)$$

where P is the probability distribution of the risk neutral probabilities p_i. Recall also from Lecture 2 that we think of the p_i's as probabilities, since for all i, $0 \leq p_i \leq 1$ and $\sum_{i=1}^{S} p_i = 1$, but the p_i's are *not* the real probabilities. In particular, p_i is *not* the probability that the economy will be in state i tomorrow.

Let P_i be the (real) probability that the economy will be in state i tomorrow. Define the *state price deflator vector* $\vec{\pi}$ in terms of the state price vector $\vec{\psi}$ s.t. $\pi_i = \psi_i / P_i$. Now

$$\begin{aligned} \vec{q} &= \psi_1 D_1 + \cdots + \psi_S D_S \\ &= P_1 \pi_1 D_1 + \cdots + P_S \pi_S D_S \end{aligned}$$

Thus,

$$\vec{q} = E^{\mathcal{P}}(\pi D) \tag{3.1}$$

where $\mathcal{P}$ is the probability distribution of the real probabilities P_i. For the remainder of this lecture, we consider expected value with respect to $\mathcal{P}$, unless otherwise stated.

3.2 Expected Excess Return

Consider an arbitrary portfolio $\vec{\theta}$ of cost \$1: *i.e.,* $\vec{q} \cdot \vec{\theta} = 1$. Let $\vec{R}^\theta \in \Re^S$ be the return on this portfolio, depending on the state of the economy:

$$\begin{aligned} \vec{R}^\theta &= (R_1^\theta, \ldots, R_S^\theta) \\ &= (D_1 \cdot \vec{\theta}, \ldots, D_S \cdot \vec{\theta}) \end{aligned}$$

Now

$$\begin{aligned} E(\pi R^\theta) &= P_1\pi_1(D_1 \cdot \vec{\theta}) + \cdots + P_S\pi_S(D_S \cdot \vec{\theta}) \\ &= (P_1\pi_1 D_1 + \cdots + P_S\pi_S D_S) \cdot \vec{\theta} \\ &= E(\pi D) \cdot \vec{\theta} \\ &= \vec{q} \cdot \vec{\theta} \qquad \{\text{by equation 9.1}\} \\ &= 1 \end{aligned}$$

Thus, under the real probability distribution, expected "deflated" return is set at \$1, regardless of the choice of portfolio $\vec{\theta}$.

Example 3.2.1 Consider the two-dimensional case in which there are only two states of the economy, S_1 and S_2, as shown in Figure 3.1.

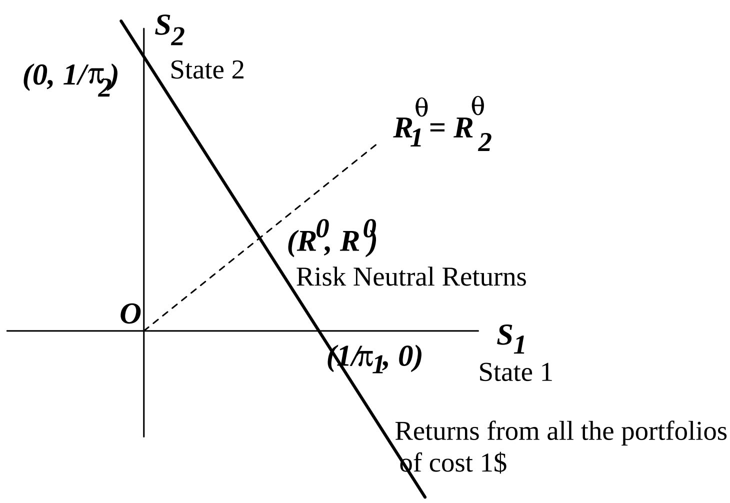


Figure 3.1: Graph of the returns obtained from a portfolio of cost 1 \$.

At any point on the x-axis, the economy is in state S_1 with probability 1; similarly, at any point on the y-axis, the economy is in state S_2 with probability 1. The solid line depicts the hyperplane satisfying $E(\pi R^\theta) = 1$. In particular, at the x-intercept, $R_1^\theta = \frac{1}{\pi_1}$ and $R_2^\theta = 0$; and, at the y-intercept, $R_1^\theta = 0$ and $R_2^\theta = \frac{1}{\pi_2}$. The dotted line is the line $R_1^\theta = R_2^\theta$. The point at intersection of these two lines is R^0, the return on the risk neutral portfolio of cost \$1. □

Now consider a risk neutral portfolio of cost \$1 with set return R^0: *i.e.*, for all i, $R_i^\theta = R^0$. For a risk neutral portfolio, we have

$$E(R^\theta) = R^0 \qquad \text{and} \qquad Var(R^\theta) = 0$$

In addition, we know from Lecture 2 that $R^0 = R$. We also have that

$$\begin{aligned} E(\pi) &= P_1\pi_1 + \cdots + P_S\pi_S \\ &= \psi_1 + \cdots + \psi_S \\ &= \frac{1}{R} \end{aligned}$$

(This last fact implies that there is a connection between the π_i's and R. We might think of π_i as $\frac{1}{R_i}$, where R_i is the discount rate in state i.)

Using these observations, we now compute the expected return on an arbitrary portfolio $\vec{\theta}$ in excess of the risk-free rate of return. Equivalently, this is the expected return in excess of the return on a risk neutral portfolio of cost \$1.

$$\begin{aligned} E(R^\theta - R^0) &= E(R^\theta) - R^0 \\ &= \frac{E(\pi)E(R^\theta) - E(\pi)R^0}{E(\pi)} \\ &= \frac{E(\pi)E(R^\theta) - (1/R)R}{E(\pi)} \\ &= \frac{E(\pi)E(R^\theta) - 1}{E(\pi)} \\ &= \frac{E(\pi)E(R^\theta) - E(\pi R^\theta)}{E(\pi)} \\ &= \frac{-E(\pi R^\theta) + E(\pi)E(R^\theta)}{E(\pi)} \\ &= \frac{-Cov(\pi, R^\theta)}{E(\pi)} \end{aligned}$$

Thus,

$$E(R^\theta - R^0) = \frac{-Cov(R^\theta, \pi)}{E(\pi)} \tag{3.2}$$

3.3 Investor Preferences

Rational behavior of investors is captured by the following two assumptions, which Markowitz incorporated into his book on modern portfolio theory in the 1950's.

1. Investors prefer more expected return to less, all other things being equal: if $E(R^{\theta_1} - R^0) > E(R^{\theta_2} - R^0)$ and $Var(R^{\theta_1}) = Var(R^{\theta_2})$, then $\vec{\theta_1}$ is preferable to $\vec{\theta_2}$.

2. Investors are averse to potential risk, all other things being equal: if $Var(R^{\theta_1}) < Var(R^{\theta_2})$ and $E(R^{\theta_1} - R^0) = E(R^{\theta_2} - R^0)$, then $\vec{\theta_1}$ is preferable to $\vec{\theta_2}$.

3.4 Capital Asset Pricing Model

The Capital Asset Pricing Model explains return on a portfolio in excess of the risk-free rate by the correlation of the portfolio with the market.

The *Sharpe Ratio* of return on a given portfolio $\vec{\theta}$ is the ratio of expected excess return to risk:

$$S(R^\theta) = \frac{E(R^\theta - R^0)}{\sigma_\theta}$$

According to our assumptions about investor preferences, investors desire to maximize this ratio, thereby maximizing expected returns and minimizing risk. Now if we normalize the Sharpe Ratio of return on portfolio $\vec{\theta}$ by the Sharpe Ratio of $\vec{\pi}$, this is equivalent to the correlation coefficient between R^θ and $\vec{\pi}$.

$$\begin{aligned}
\frac{S(R^\theta)}{S(\pi)} &= \frac{E(R^\theta - R^0)/\sigma_\theta}{E(\pi - R^0)/\sigma_\pi} \\
&= \frac{-Cov(R^\theta, \pi)/E(\pi)\sigma_\theta}{-Var(\pi)/E(\pi)\sigma_\pi} \qquad \{\text{by equation 3.2}\} \\
&= \frac{Cov(R^\theta, \pi)}{\sigma_\theta \sigma_\pi} \\
&= Corr(R^\theta, \pi)
\end{aligned}$$

In these terms, the goal of investors is to maximize the correlation between the return on portfolio $\vec{\theta}$ and $\vec{\pi}$. The portfolio $\vec{\theta}^*$ which maximizes this correlation is called the *market portfolio*:

$$\vec{\theta}^* = \arg\sup_\theta Corr(R^\theta, \pi)$$

Let R^* be the return on the market portfolio.

The Single Index Model assumes that returns are generated by a linear model:

$$R^\theta = \alpha_\theta + \beta_\theta R^* + \epsilon$$

Intuitively, returns depend only on one factor, R^*; the term $\beta_\theta R^*$ captures the systematic risk of portfolio $\vec{\theta}$ and ϵ is an error term that is uncorrelated with market return.

Given this definition of R^θ, we compute $Cov(R^\theta, \pi)$ and $Cov(R^\theta, R^*)$.

$$\begin{aligned} Cov(R^\theta, \pi) &= \alpha_\theta E(\pi) + \beta_\theta E(R^*\pi) - \alpha_\theta E(\pi) - \beta_\theta E(R^*)E(\pi) \\ &= \beta_\theta E(R^*\pi) - \beta_\theta E(R^*)E(\pi) \\ &= \beta_\theta Cov(R^*, \pi) \end{aligned}$$

$$\begin{aligned} Cov(R^\theta, R^*) &= \alpha_\theta E(\pi) + \beta_\theta E(R^{*2}) - \alpha_\theta E(\pi) - \beta_\theta E(R^*)^2 \\ &= \beta_\theta E(R^{*2}) - \beta_\theta E(R^*)^2 \\ &= \beta_\theta Var(R^*) \end{aligned}$$

Now consider the ratio of expected excess return on portfolio $\vec{\theta}$ to expected excess return on the market portfolio. This ratio is exactly β_θ.

$$\frac{E(R^\theta - R^0)}{E(R^* - R^0)} = \frac{Cov(R^\theta, \pi)}{Cov(R^*, \pi)} = \beta_\theta$$

Also, by the above,

$$\beta_\theta = \frac{Cov(R^\theta, R^*)}{Var(R^*)}$$

"*Stick that in your pipe and smoke it.*" (Even, '95)

Chapter 4

Scribe: Marek Teichmann, April 13, 1995

4.1 The Multiperiod model

As before, we have N assets, and similar dividend processes. In the single period model, we go from the current state to the next state, for example either up or down. In the multiperiod model, this happens several times. We have a following picture, called the *binomial tree*, see figure 4.1. Each node represents a state and in column T, the possible states reachable after T transitions, at which point there are $T+1$ different states. From the initial state to a state at time T there are $\binom{T+1}{\frac{T+1}{2}}$ possible paths. If it is possible to go neither up or down, i.e. stay at the same level, we get horizontal edges and the diagram is called the *trinomial tree*. We can think of a path in this tree as a random walk on this lattice. If the time scale is small enough, this model with approximate the real situation well.

Let N be the number of assets, and $0 \leq t \leq T$ be a point in time. We have two stochastic processes: A *dividend process*

$$\vec{\delta}_t = (\delta_t^1, \delta_t^2, \ldots, \delta_t^N).$$

At time t we get paid δ_t^i for asset i. We also have a *price process*

$$\vec{S}_t = (S_t^1, S_t^2, \ldots, S_t^N).$$

So for a particular path in the binomial tree, we have a set of prices and dividends given by the above vectors.

We now must come up with a *trading strategy*:

$$\vec{\theta}_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^N).$$

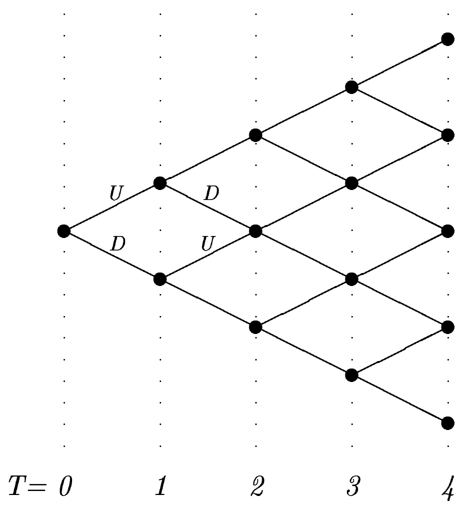


Figure 4.1: Binomial tree

At time t we hold θ_t^i amount of asset i and so on. $\vec{\theta_t}$ is a function of δ and S, but depends only on the past, not the future. This called an *adapted process*. We will assume that our trading strategy is an adapted process[1].

4.1.1 Trading strategy

Our trading strategy can be described as follows:

1. Start with a portfolio $\theta_{-1} = 0$ of zero cost.

2. At time $t = 0$ we create a portfolio θ_0.

3. At time $t \geq 1$, we buy and sell going from θ_{t-1} to θ_t, and we get the dividend generated at time t:

$$\delta_t = \vec{\theta}_{t-1} \cdot \vec{\delta}_t + \left(\vec{\theta}_{t-1} - \vec{\theta}_t\right) \vec{S}_t$$

Where the first product is the dividend obtained from the portfolio held at time $t-1$, and the second is due to the change in the portfolio, where we buy and sell at the price at time t.

[1]Formally: $\vec{\theta_t} \in \sigma(\delta, S)$

4.1.2 Stochastic processes

A stochastic process $X_t = (\vec{S}_t, \vec{\delta}_t)$ generates $\vec{S}_t$ and $\vec{\delta}_t$ at time t. It satisfies

$$Pr[X_t \in A | X_0 = x_0, \ldots, X_{t-1} = x_{t-1}] = Pr[X_t \in A | X_{t-1} = x_{t-1}]$$

We can take any particular sequence which gives us a path. An event is a finite collection of paths.

A filtration $\mathcal{F}_t = \{X_0 = x_0, \ldots, X_t = x_t\}$ is the set of all paths of this type.

An adapted process is $Y_t \in \mathcal{F}_t$. What happens at time t depends only on $x_0, \ldots, x_t$.

4.1.3 No arbitrage

Then the no arbitrage condition is equivalent to:

$$(\forall x \in X)\,(\forall 0 \le t \le T)\left[\delta_t^\theta(x) \ge 0\right] \longrightarrow (\forall x \in X)\,(\forall 0 \le t \le T)\left[\delta_t^\theta(x) = 0\right]$$

i.e. there is no scheme s.t. no matter what the adversary does, one makes money or breaks even. This is again equivalent to

$$\neg\,(\forall x \in X)\left[\{\delta_0^\theta(x), \ldots, \delta_T^\theta(x)\} \notin \mathbb{R}_{\ge 0}^{T+1} \setminus \{0\}\right]$$

We can write the price at time t as

$$S_t = \frac{1}{\pi_t} E_t \left(\sum_{j=t+1}^{T} \pi_j \delta_j \right) \qquad 0 \le t \le T \tag{4.1}$$

This will be shown below. Hence

$$S_t \in \text{Interior PositiveHull}(\delta_{t+1}, \ldots, \delta_T)$$

We write the expectations as E_t since the probabilities of going up or down are time dependent.

Let L be the space of adapted processes. Then $\theta \in L$ and $\delta^\theta \in L$ are also adapted processes.

Let $L_+ = \{x \ge 0 | x \in L\}$. Then $L_+ \setminus \{0\}$ is the set of dividend processes corresponding to arbitrage.

Let M be the set of dividend processes, the marketed subspace. Then we have no arbitrage if and only if $M \cap L_+ = \{0\}$.

The situation is depicted in figure 4.1.3.

Lemma 4.1.1 *M is a linear subspace.*

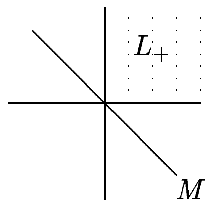


Figure 4.2: L_+ and M intersect only at 0

PROOF
Let θ and $\phi \in L$. Then $a\delta^\theta + b\delta^\phi = \delta^{a\theta + b\phi}$.

$$\begin{aligned}\delta^{a\theta+b\phi} &= a\vec{\theta}_{t-1}\delta_t + a(\vec{\theta}_{t-1} - \vec{\theta}_t)S_t + \\ &\quad b\vec{\phi}_{t-1}\delta_t + b(\vec{\phi}_{t-1} - \vec{\phi})S_t \\ &= a\delta_t^\theta + b\delta_t^\phi.\end{aligned}$$

Theorem 4.1.2 (Reeds representation theorem) *$\exists F : L \longrightarrow \mathbb{R}$ s.t. Ker $F = M$.*

So there is no arbitrage iff there exists an $F : L \longrightarrow \mathbb{R}$, F linear, strictly increasing such that

$$F(\delta^\theta) = 0, \; F(\theta) = E_t\left(\sum_{t=0}^{T} \pi_t \theta_t\right) \;\; \pi_t > 0.$$

If there was arbitrage, M would move away from the origin towards the interior of L_+, and would create a region where all coordinates are positive.

$$\begin{aligned}F(\delta^\theta) &= E_t\left(\sum_{t=0}^{T} \pi_t \delta_t^\theta\right) \\ &= E_t\left[\sum_{t=0}^{T} \pi_t \theta_{t-1}\delta_t + \sum_{t=0}^{T} \pi_t(\theta_{t-1} - \theta_t)S_t\right] = 0\end{aligned}$$

We now choose one adapted strategy:

$$\theta_{-1} = \theta_0 = \ldots = \theta_{t-1} = 0$$

and some arbitrary $\hat{\theta} = \theta_t = \ldots = \theta_T \in \mathbb{R}^N$. We enter the market at time t with arbitrary portfolio $\hat{\theta}$ and hold it to the end. This sum then becomes

$$F(\delta^\theta) = E_t\left[\sum_{j=t+1}^{T} \pi_j \hat{\theta}\delta_j) - \pi_t \hat{\theta} S_t\right]$$

We can now write

$$\pi_t \hat{\theta} S_t = \hat{\theta} E_t \left[\sum_{j=t+1}^{T} \pi_j S_j \right] \Longrightarrow S_t = \frac{1}{\pi_t} E_t \left[\sum_{j=t+1}^{T} \pi_j S_j \right]$$

We have proved (4.1).

We can also derive the market value of a trading strategy as follows[2]

$$\begin{aligned}
\theta_{t-1} S_{t-1} &= \frac{1}{\pi_{t-1}} E \left[\pi_t \theta_{t-1} \delta_t + \theta_{t-1} \sum_{j=t+1}^{T} \pi_j \delta_j \right] \\
&= \frac{1}{\pi_{t-1}} E \left[\pi_t \theta_{t-1} \delta_t + \theta_{t-1} \pi_t S_j \right] \\
&= \frac{\pi_t}{\pi_{t-1}} \left(E(\theta_{t-1} \delta_t) + \theta_{t-1} S_t \right)
\end{aligned}$$

Hence

$$\begin{aligned}
\theta_{t-1} S_{t-1} - \frac{\pi_t}{\pi_{t-1}} \theta_t S_t &= \frac{\pi_t}{\pi_{t-1}} \left(E(\theta_{t-1} S_t) + (\theta_{t-1} - \theta_t) S_t \right) \\
&= \frac{\pi_t}{\pi_{t-1}} E(\delta_t^{\theta})
\end{aligned}$$

So

$$\pi_{t-1} \theta_{t-1} S_{t-1} - \pi_t \theta_t S_t = E(\pi_t \delta_t^{\theta})$$

We assume that at the end all prices drop to zero:

$$\pi_{T-1} \theta_{T-1} S_{T-1} - \underbrace{\pi_T \theta_T S_T}_{0} = E(\pi_T \delta_T^{\theta})$$

By telescoping and linearity of expectation, we get

$$\theta_t S_t = \frac{1}{\pi_t} E\left(\sum_{j=t+1}^{T} \pi_j \delta_j^{\theta} \right)$$

Where $\theta_t S_t$ is the market value of a trading strategy at time t and is equal to the expected dividends divided by a discounting factor. Going back to the binomial tree, we can determine the value at the root, given the dividends and the discounting factors at each stage, by calculating the prices at the different states.

[2] Here we ignore the fact that the probabilities themselves change at each stage.

Chapter 5

Scribe: Ron Even, April 20, 1995

5.1 Probability Space

Let's begin by defining a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where Ω is a sample space and $\mathcal{F}$ is a $\sigma - field$ over Ω, i.e.

$$
\begin{array}{lll}
\emptyset & \in & \mathcal{F} \\
\Omega & \in & \mathcal{F} \\
\mathcal{F} & & is\ closed\ under\ countable\ union,\ complement,\ and\ intersection.
\end{array}
$$

Let $\mathcal{X}$ be a Markovian price process, $\mathcal{X} = \{x_0, x_1, \ldots, x_T\}$, i.e. $Pr[x_t \in A | x_0, \ldots, x_{t-1}] = Pr[x_t \in A | x_{t-1}]$. It is of no interest to look at probabilities of individual paths $\{x_0, \ldots, x_T\}$ since these all have probability zero; therefore, each event is a collection of paths, $\mathcal{F}_t = \{x_0 \in A_0, x_1 \in A_1, \ldots, x_t \in A_t\}, A_i \subseteq \Omega$. With events so defined, we can examine the probability that a price stays within some range, crosses some barrier, etc. We have a filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \ldots$. We define an adapted trading strategy $\{Y_0, Y_1, \ldots, Y_T\}$ where

$$
\begin{array}{rcl}
Y_t & = & \Phi_t(x_0, x_1, \ldots, x_T) \\
\forall j, \Phi_j(x) & is & \mathcal{F}_j - measurable \\
\{Y_j = \Phi_j(x) \leq \alpha\} & \in & \mathcal{F}_j
\end{array}
$$

Theorem 1 *There is no arbitrage* $\Leftrightarrow \forall t, \pi_t \theta_t S_t = E[\sum_{j=t+1}^{T} \delta_j^{\theta} \pi_j | \mathcal{F}_t]$.

Let's examine the one day portfolio, $\theta_t = \bar{\theta} = constant, \theta_{t+1} = \ldots = \theta_T = 0$.

$$\pi_t \bar{\theta} S_t = E[\bar{\theta} \pi_{t+1} S_{t+1} | \mathcal{F}_t]$$

Suppose it was the case that $S_t = E[S_{t+1}|\mathcal{F}_t]$, i.e. today's price is the expected value of tomorrow's price. We call x_t a martingale if $x_t = E[x_{t+1}|x_0, x_1, \ldots, x_t]$.

$$\pi_t S_t = E[\pi_{t+1} S_{t+1}|\mathcal{F}_t]$$

The price process multiplied by the deflator is a martingale. Now let's examine a riskless asset (e.g. a bond).

$$\begin{aligned} 1 &= E[\frac{\pi_{t+1}}{\pi_t} R|\mathcal{F}_t] \\ S_t &= \frac{1}{R} E[\frac{\pi_{t+1}}{\pi_t} R S_{t+1}|\mathcal{F}_t] \\ S_t &= \frac{1}{R} E[S_{t+1} E[\frac{\pi_{t+1}}{\pi_t}|\mathcal{F}_t]|\mathcal{F}_t] \\ S_t &= \frac{1}{R} E^Q[S_{t+1}|\mathcal{F}_t] \end{aligned}$$

In the last line above, we've adjusted the probability space E^Q so that we no longer need the price deflators.

$$\begin{aligned} S_t &= E^Q[\frac{S_{t+1}}{R^{t,t+1}}|\mathcal{F}_t] \\ \frac{S_t}{R_{0,t}} &= E^Q[\frac{S_t}{R_{0,T}}|\mathcal{F}_t] \\ R_{t,T} &= \frac{R_{0,T}}{R_{0,t}} \end{aligned}$$

In the above equations, $\frac{S_t}{R_{0,t}}$ is the discounted price process.

5.2 Some examples of martingales

- Random Walk: $x_{n+1} = x_n \pm b$ where b is Bernoulli with $Pr = 0.5$.
- Brownian Motion: $x_{n+1} = x_n + \sigma n$ where σ is Gaussian normally distributed with mean 0 and standard deviation 1.
- Multiplicative Random Walk: $x_{n+1} = x_n \xi$ where

$$\xi = \begin{cases} U & \text{with probability } p_U \\ D & \text{with probability } p_D \end{cases}$$

$$E[x_{n+1}] = E[\xi x_n] = x_n(p_U U + p_D D) = x_n$$

with the last equality coming by the definition of a martingale. Solving these equalities, we arrive at $p_D = \frac{1-U}{D-U}$ and $p_U = \frac{D-1}{D-U}$. The multiplicative random walk is preferred to the random walk since it does not allow negative prices to arise and is therefore more realistic.

Chapter 6

Scribe: Amy Greenwald, May 11, 1995

6.1 Review

We assume the usual processes:

- a *price process* $\{S_0, \ldots, S_T\}$, where S_t is the price of asset S at time t
- a *discount process* $\{R_0, \ldots, R_T\}$, where R_t is the rate of return on a risk-free asset of value 1 over period $[t, t+1)$: *i.e.:* R_t is the short-term interest rate
- a *discounted price process*

$$\{S_0, \frac{S_1}{R_{0,1}}, \ldots, \frac{S_T}{R_{0,T}}\}$$

  (Notation: $R_{t_1,t_2} \equiv R_{t_1} R_{t_1+1} \ldots R_{t_2}$ over the period $[t_1, t_2)$.)

- a *state price deflator process* $\{\pi_0, \ldots, \pi_T\}$, where π_t is the state price deflator at time t

Given these processes, we restate some facts shown in previous lectures.

Fact 1 *Note the following:*

$$\pi_t = E\{\pi_{t+1} R_t \mid \mathcal{F}_t\}$$

By iterating this equation, we arrive at:

$$\pi_t = E\{\pi_\tau R_{t,\tau} \mid \mathcal{F}_t\}, \quad \tau > t$$

Fact 2 *The discounted price process is a Martingale:* i.e.:

$$\frac{S_t}{R_{0,t}} = E^{\mathcal{Q}}\left\{\frac{S_T}{R_{0,T}} \mid \mathcal{F}\right\}$$

Fact 3 *The no arbitrage condition in a multiperiod model implies:*

$$\pi_t \theta_t S_t = E\left\{\sum_{j=t+1}^{T} \pi_j \delta_j^\theta \mid \mathcal{F}_t\right\} \tag{6.1}$$

We rewrite equation 6.1 as follows:

$$\theta_t S_t = E\left\{\sum_{j=t+1}^{T} \left(\frac{\delta_j^\theta}{R_{t,j}}\right)\left(\frac{\pi_j R_{t,j}}{\pi_t}\right) \mid \mathcal{F}_t\right\}$$

and again as:

$$\theta_t S_t = E^{\mathcal{Q}}\left\{\sum_{j=t+1}^{T} \left(\frac{\delta_j^\theta}{R_{t,j}}\right) \mid \mathcal{F}_t\right\}$$

where $\mathcal{Q}$ is the Martingale measure (or the pricing measure) *i.e.:* some new probability measure based on the $\pi_t s$. Thus, the value of a portfolio is equal to the sum over the time to maturity of the expected value (under the appropriate distribution) of the discounted dividends, given the information available today.

6.2 Binomial Option Pricing Model

The binomial model of option pricing is a discrete approximation to the Black-Scholes equations.

Let

$$S_{t+1} = S_t \xi,$$

where

$$\xi = \begin{cases} U & \text{with probability } p_U \\ D & \text{with probability } p_D \end{cases}$$

Fix $R_t = R$.

The discounted price process is a Martingale; thus,

$$S_t = E^{\mathcal{Q}}\left\{\frac{S_{t+1}}{R} \mid \mathcal{F}_t\right\}$$

which implies:

$$S_t = \frac{1}{R}(p_U SU + p_D SD)$$

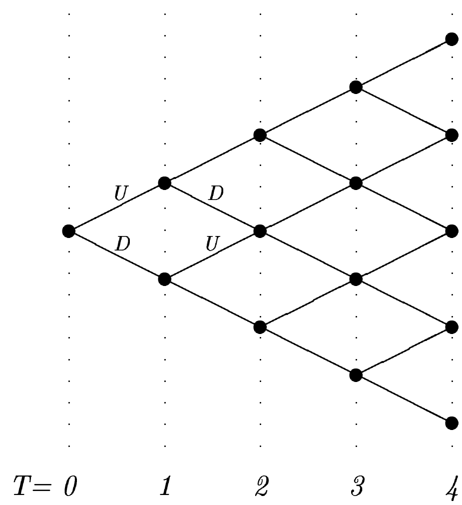


Figure 6.1: Binomial tree

or equivalently,

$$R = p_U U + p_D D$$

Also, since p_U and p_D are probability measures:

$$1 = p_U + p_D$$

Solving this system of equations yields:

$$p_U = \frac{R - D}{U - D} \qquad p_D = \frac{U - R}{U - D}$$

We now derive a formula to value European calls:

At the end of T periods, the asset S can be valued at one of $T + 1$ possible prices:

$$SU^T, SU^{T-1}D, \ldots, SUD^{T-1}, SD^T$$

The probability that the price is $SU^j D^{T-j}$, for $0 \leq j \leq T$, is given by the Binomial Theorem:

$$\left(\begin{array}{c} T \\ j \end{array} \right) p_U^j p_D^{T-j}$$

Now if C_0 denotes the current value of a call option on an underlying asset with price S, then

$$\begin{aligned} C_0 &= E^{\mathcal{Q}} \left\{ \tfrac{f(S_T)}{R^T} \mid \mathcal{F}_t \right\} \\ &= \tfrac{1}{R^T} \textstyle\sum_{j=0}^T \left(\begin{array}{c} T \\ j \end{array} \right) p_U^j p_D^{T-j} f(SU^j D^{T-j}) \\ &= \tfrac{1}{R^T} \textstyle\sum_{j=0}^T \left(\begin{array}{c} T \\ j \end{array} \right) \left(\tfrac{R-D}{U-D} \right)^j \left(\tfrac{U-R}{U-D} \right)^{T-j} f(SU^j D^{T-j}) \end{aligned}$$

In the above derivation, f is just some function of the price of the underlying asset. In the next lecture we will refine this derivation by explicitly defining f as a function of the asset price and the strike price.

Chapter 7

Scribe: Amy Greenwald, June 8, 1995

7.1 Binomial Option Pricing Model

In this lecture we continue the discussion of the binomial option pricing model to value European calls. Let C_t be the value at time t of a call option on a stock with price S_t. The value C_t is equal to the expected payoffs, discounted by the time til expiration, given the information available at time t. If T is the expiration date, this is expressed formally as:

$$C_t = E^Q \left\{ \frac{C_T}{R^{T-t}} \mid \mathcal{F}_t \right\}$$

Let K be the strike price. The value C_T of the call option at time T is given by the payoffs:

$$C_T = \begin{cases} S_T - K & \text{if } S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

i.e., $C_T = \max\{S_T - K, 0\}$, which we abbreviate as $(S_T - K)^+$.

Recall from last time that the discounted price process is a Martingale; it follows that the probabilities p_U and p_D are given by:

$$p_U = \frac{R-D}{U-D} \qquad\qquad p_D = \frac{U-R}{U-D}$$

with $D \leq R \leq U$, since there is no arbitrage. We now refine the derivation given last time of the value C_0 of a call option at time 0:

$$\begin{aligned} C_0 &= \tfrac{1}{R^T} E^Q\{(S_T - K)^+ \mid \mathcal{F}_t\} \\ &= \tfrac{1}{R^T} \sum_{j=0}^{T} \begin{pmatrix} T \\ j \end{pmatrix} p_U^j p_D^{T-j} (S_0 U^j D^{T-j} - K)^+ \\ &= \tfrac{S_0}{R^T} \sum_{j=0}^{T} \begin{pmatrix} T \\ j \end{pmatrix} \left(\tfrac{R-D}{U-D}\right)^j \left(\tfrac{U-R}{U-D}\right)^{T-j} (U^j D^{T-j} - \tfrac{K}{S_0})^+ \end{aligned}$$

Now find the maximum j *s.t.*

$$U^j D^{T-j} \leq \frac{K}{S_0}$$

i.e.,

$$\left(\frac{U}{D}\right)^j D^T \leq \frac{K}{S_0}$$

Algebraic manipulation yields:

$$j^* = \frac{\log K - \log S_0 - T \log D}{\log(U/D)}$$

Thus,

$$\begin{aligned} C_0 = \; & S_0 \sum_{j=j^*}^{T} \begin{pmatrix} T \\ j \end{pmatrix} \left(\tfrac{R-D}{U-D}\right)^j \left(\tfrac{U-R}{U-D}\right)^{T-j} \left(\tfrac{U}{D}\right)^j \left(\tfrac{D}{R}\right)^T - \\ & \tfrac{K}{R^T} \sum_{j=0}^{T} \begin{pmatrix} T \\ j \end{pmatrix} \left(\tfrac{R-D}{U-D}\right)^j \left(\tfrac{U-R}{U-D}\right)^{T-j} \end{aligned}$$

This equation implies that a call option behaves like a portfolio of stocks and bonds in which we buy stocks worth the price of the underlying asset S_0 and sell bonds worth the strike price K. Note that a future is a special kind of call option where $K = 0$.

7.2 Black-Scholes Model

In the next lecture, we derive the continuous-time analogue of the binomial option pricing model, which is known as the Black-Scholes equations:

$$\begin{aligned} C_0 &\approx S_0 \int_{d-\sigma\sqrt{t}}^{\infty} \tfrac{e^{-y^2/2}}{\sqrt{2\pi}} dy - \tfrac{K}{R^T} \int_{d}^{\infty} \tfrac{e^{-y^2/2}}{\sqrt{2\pi}} dy \\ &= N(-d + \sigma\sqrt{t}) * \text{ STOCK PRICE } - \\ &\quad\; N(-d) * \text{ DISCOUNTED BOND PRICE} \end{aligned}$$

For the moment, let us restrict our attention to one stage in the binomial tree:

$$\begin{aligned} C_t &= \tfrac{1}{R} E^Q\{C_{t+1} \mid \mathcal{F}_t\} \\ &= \tfrac{1}{R}(C_{t+1}^U p_U + C_{t+1}^D p_D) \end{aligned}$$

In fact, C_t depends on S; thus, we have a sequence of difference equations

$$C_t(S) = \frac{1}{R_t}\left[C_{t+1}(SU)\left(\frac{R_t - D}{U - D}\right) + C_{t+1}(SD)\left(\frac{U - R_t}{U - D}\right)\right], 0 \le t < T$$

with boundary condition:

$$C_T(S) = (S - K)^+$$

Now, if we consider time steps of size Δt, this yields a set of differential equations

$$C_t(S) = \frac{1}{R_t}\left[C_{t+\Delta t}(SU)\left(\frac{R_t - D}{U - D}\right) + C_{t+\Delta t}(SD)\left(\frac{U - R_t}{U - D}\right)\right], 0 \le t < T$$

with the appropriate boundary condition. These equations have solution

$$\frac{-\partial C}{\partial t} = \frac{\sigma^2 S^2}{2}\ \frac{\partial^2 C}{\partial S^2} + rS\ \frac{\partial C}{\partial S} - rC$$

Thus, the negative of the rate of change of the value of the call option with respect to time depends only on the interest rate r and the volatility σ of the stock, but not on the return μ of the stock.

Note the Black Scholes equations are based on the assumption that the bond and the stock prices evolve according to the processes described by the following diffusion equations, respectively:

$$\frac{dB}{B} = r\ dt$$

$$\frac{dS}{S} = \mu\ dt + \sigma\ dW$$

7.3 Replication

In this section, we construct a hedge fund consisting of stocks and bonds that is equivalent in value to a call option. In what follows, $\{\Delta_t\}$ is an adapted process that is adjusted over time by buying and selling stock. In particular, Δ_t is the amount of stock held at time t:

$$C_t = \Delta_t S_t + B_t$$

We choose Δ_t such that if the market goes up, then:

$$C_{t+1}(S_t U) = \Delta_t S_t U + B_t R_t$$

otherwise, if the market goes down, then:

$$C_{t+1}(S_t D) = \Delta_t S_t D + B_t R_t$$

Solving for Δ_t yields:

$$\Delta_t = \frac{C_{t+1}(S_tU) - C_{t+1}(S_tD)}{S_t(U-D)}$$

Thus,

$$B_t = \frac{1}{R_t}\left(\frac{UC_{t+1}(S_tD) - DC_{t+1}(S_tU)}{U-D}\right)$$

Finally, we note that this is in fact a replicating portfolio:

$$\begin{aligned}
\Delta_t S_t + B_t &= \tfrac{C_{t+1}(S_tU) - C_{t+1}(S_tD)}{U-D} + \tfrac{1}{R_t}\left(\tfrac{UC_{t+1}(S_tD) - DC_{t+1}(S_tU)}{U-D}\right) \\
&= \tfrac{1}{R_t}\left[C_{t+1}(S_tU)\left(\tfrac{R_t - D}{U-D}\right) + C_{t+1}(S_tD)\left(\tfrac{U-R_t}{U-D}\right)\right] \\
&= C_t
\end{aligned}$$

Chapter 8

Scribe: Ron Even, June 15, 1995

8.1 The Black-Scholes Equation

8.1.1 European Call Option

Let's begin by looking at the expected price of a European call option with strike price K and expiration at time T.

$$\begin{aligned} C_0^T &= E^Q[(S_T - K)^+ R^{-T} | \mathcal{F}_l] \\ &= R^{-T} \sum_{j=0}^{T} \binom{T}{j} p_U^j p_D^{T-j} (S_0 U^j D^{T-j} - K)^+ \end{aligned}$$

Up until now we have studied a discrete, multi-period model. Now we will move to a continuous time model. We break up each discrete interval into N sub-intervals, and examine our system as $N \to \infty$; thus, if we look at the duration of each sub-interval, we have $\frac{1}{N} \to 0$. For now, we will look at yearly intervals.

8.1.2 Continuous Finance

$$\begin{aligned} \mu &= \text{annual yearly return on stock } S \\ \sigma &= \text{annual volatility of stock } S \\ r &= \text{annual interest rate} \\ R &= 1 + \frac{r}{N} \\ U &= e^{\frac{\mu}{N} + \frac{\sigma}{\sqrt{N}}} \\ D &= e^{\frac{\mu}{N} - \frac{\sigma}{\sqrt{N}}} \end{aligned}$$

Thus, the equation for Brownian motion becomes,

$$\frac{dS}{S} = \mu dt + \sigma dW$$

Substituting into the equations for U and D the formula $e^x = 1 + x + \frac{x^2}{2!} + \cdots$ gives us

$$\begin{aligned} U &= 1 + \frac{\mu}{N} + \frac{\sigma}{\sqrt{N}} + \frac{1}{2}(\frac{\mu}{N} + \frac{\sigma}{\sqrt{N}})^2 + o(\frac{1}{N}) \\ &= 1 + \frac{\mu}{N} + \frac{\sigma}{\sqrt{N}} + \frac{1}{2}\frac{\sigma^2}{N} + o(\frac{1}{N}) \\ D &= 1 + \frac{\mu}{N} - \frac{\sigma}{\sqrt{N}} + \frac{1}{2}\frac{\sigma^2}{N} + o(\frac{1}{N}) \end{aligned}$$

Thus, $U - D \approx \frac{2\sigma}{\sqrt{N}}$. Substituting our approximations for U and D back into the equations for p_U and p_D from the previous lesson, we arrive at

$$\begin{aligned} p_U &= \frac{R-D}{U-D} \\ &= \frac{1}{\frac{2\sigma}{\sqrt{N}}}[1 + \frac{r}{N} - 1 - \frac{\mu}{N} + \frac{\sigma}{\sqrt{N}} - \frac{\sigma^2}{2N}] \\ &= \frac{1}{2} + \frac{1}{2\sigma\sqrt{N}}(r - \mu - \frac{\sigma^2}{2}) \\ p_D &= \frac{U-R}{U-D} \\ &= \frac{1}{2} - \frac{1}{2\sigma\sqrt{N}}(r - \mu - \frac{\sigma^2}{2}) \end{aligned}$$

Next, we look at the variable ξ defined by

$$\xi = \begin{cases} +1 & \text{with probability } p_U \\ -1 & \text{with probability } p_D \end{cases}$$

$$\begin{aligned} E(\xi) &= (+1)p_U + (-1)p_D \\ &= \frac{1}{\sigma\sqrt{N}}(r - \mu - \frac{\sigma^2}{2}) \\ E(\xi^2) &= (+1)^2 p_U + (-1)^2 p_D \\ &= 1 \\ Var(\xi) &= E(\xi^2) - E(\xi)^2 \\ &= 1 - \frac{1}{\sigma^2 N}(r - \mu - \frac{\sigma^2}{2})^2 \end{aligned}$$

Thus, we can write $\xi = \frac{1}{\sigma\sqrt{N}}(r-\mu-\frac{\sigma^2}{2})+\xi'$ where ξ' is Bernoulli distributed with mean 0 and $Var(\xi') = 1 - \frac{1}{\sigma^2 N}(r-\mu-\frac{\sigma^2}{2})$. Note that as $N \to \infty$, $Var(\xi') \to 1$. As a consequence of the Central Limit Theorem, if we add K such independent identically distributed random variables, we get $\sim \sqrt{K}N(0,1)$.

8.1.3 Calculating Final Return

Let S_i be our stock price at time i. We start at S_0 and would like to know the price at time NT, after T intervals.

$$\begin{aligned}
S_{NT} &= S_0 e^{\frac{\mu}{N}+\frac{\sigma}{\sqrt{N}}\xi_1} e^{\frac{\mu}{N}+\frac{\sigma}{\sqrt{N}}\xi_2} \cdots e^{\frac{\mu}{N}+\frac{\sigma}{\sqrt{N}}\xi_{NT}} \\
&= S_0 (e^{\frac{\mu}{N}})^{NT} e^{\frac{\sigma}{\sqrt{N}}\sum_{i=1}^{NT}\xi_i} \\
&= S_0 e^{\mu T} e^{\frac{\sigma}{\sqrt{N}}[NT\frac{1}{\sigma\sqrt{N}}(r-\mu-\frac{\sigma^2}{2})+\sum_{i=1}^{NT}\xi_i']} \\
&= S_0 e^{\mu T} e^{T(r-\mu-\frac{\sigma^2}{2})} e^{\frac{\sigma}{\sqrt{N}}\sqrt{NT}N(0,1)} \\
&= S_0 e^{\mu T} e^{T(r-\mu-\frac{\sigma^2}{2})} e^{\sigma\sqrt{T}N(0,1)} \\
&= S_0 e^{(r-\frac{\sigma^2}{2})T} e^{\sigma\sqrt{T}N(0,1)}
\end{aligned}$$

One important observation here is that the final return does not depend upon μ, the annual return of the stock. This is to be expected, as our no arbitrage condition guarantees that the return must be determined by the volatility of the stock and the risk-free interest rate.

8.1.4 Calculating the Value of the Call Option

The value of the call option, C_0^T is $R^{-NT}E^Q[(S_{NT}-K)^+]$. The first step in calculating this is to calculate the expected value of S_{NT}.

Expected Value of S_{NT}

$$\begin{aligned}
E(S_{NT}) &= S_0 e^{(r-\frac{\sigma^2}{2})T} \int_{-\infty}^{\infty} e^{\sigma\sqrt{T}} \frac{e^{\frac{-y^2}{2}}}{\sqrt{2}\pi} dy \\
&= S_0 e^{(r-\frac{\sigma^2}{2})T} \int_{-\infty}^{\infty} \frac{e^{\frac{-y^2-2\sigma\sqrt{T}+\sigma^2 T}{2}}}{\sqrt{2}\pi} e^{\frac{\sigma^2 T}{2}} dy \\
&= S_0 e^{(r-\frac{\sigma^2}{2})T} e^{\frac{\sigma^2 T}{2}} \int_{-\infty}^{\infty} \frac{e^{\frac{-(y^2-\sigma\sqrt{T})^2}{2}}}{\sqrt{2}\pi} dy \\
&= S_0 e^{(r-\frac{\sigma^2}{2})T} e^{\frac{\sigma^2 T}{2}} \\
&= S_0 e^{rT}
\end{aligned}$$

Value of the Call Option

Using our results from the previous section, we get:

$$
\begin{aligned}
C_0^T &= R^{-NT} E^Q[(S_{NT} - K)^+] \\
R^{-NT} &= ((1 + \frac{r}{N})^{\frac{N}{r}})^{-rT} \\
&= e^{-rT} \\
C_0^T &= e^{-rT} \int_{-\infty}^{\infty} (S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}y} - K)^+ \frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}} dy
\end{aligned}
$$

We need to find max(d) such that $S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}d} \leq K$.

$$
\begin{aligned}
\log(K) &= \log(S_0) + (r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}d \\
d &= \frac{\log(K) - \log(S_0) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}
\end{aligned}
$$

Now we can continue finding a formula for C_0^T.

$$
\begin{aligned}
C_0^T &= e^{-rT} \int_d^{\infty} S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}y} \frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}} dy - K e^{-rT} \int_d^{\infty} \frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}} dy \\
&= S_0 e^{\frac{-\sigma^2}{2}T} \int_d^{\infty} \frac{e^{\frac{-y^2 - 2\sigma\sqrt{T}y + \sigma^2 T}{2}}}{\sqrt{2\pi}} e^{\sigma^2 T} dy - K e^{-rT} \int_d^{\infty} \frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}} dy \\
&= S_0 \int_d^{\infty} \frac{e^{\frac{-(y-\sigma\sqrt{T})^2}{2}}}{\sqrt{2\pi}} dy - K e^{-rT} \int_d^{\infty} \frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}} dy \\
&= S_0 \int_{d-\sigma\sqrt{T}}^{\infty} \frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}} dz - K e^{-rT} \int_d^{\infty} \frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}} dy \\
&= S_0 \Phi[-d + \sigma\sqrt{T}] - K e^{-rT} \Phi[-d] \text{ where} \\
\Phi[x] &= \int_{-\infty}^{x} \frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}} dy \\
&= \int_{-x}^{\infty} \frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}} dy
\end{aligned}
$$

Chapter 9

Scribe: Amy Greenwald, June 22, 1995

9.1 Brownian Motion

In the second half of this lecture, we give an alternative derivation of the Black-Scholes equations utilizing the Ito Calculus. We begin by describing Brownian motion as the limit of a binomial process - in particular, as the limit of a random walk.

Let $X_1, X_2, \ldots, X_N, \ldots$ denote a sequence of Bernoulli random variables that are pairwise independent and take values of $+1$ and -1. A random walk (S_N) is defined by:

$$\begin{aligned} S_0 &= 0 \\ S_N &= \textstyle\sum_{i=1}^{N} X_i \end{aligned}$$

By the Central Limit Theorem, $(S_N) \sim N(0, \sqrt{N})$. Now by fixing a time t and interpolating as follows:

$$W_N(t) = \frac{1}{\sqrt{N}} S_{\lfloor Nt \rfloor} + \frac{Nt - \lfloor Nt \rfloor}{\sqrt{N}} (S_{\lfloor Nt+1 \rfloor} - S_{\lfloor Nt \rfloor})$$

we obtain, in the limit as $N \to \infty$,

$$W(t) = \lim_{N \to \infty} W_N(t)$$

$W(t)$ is a Brownian motion with drift 0 and diffusion 1.

Brownian motion exhibits several important properties which justify its use as a modelling tool in continuous-time finance.

1. $W(t) \sim N(0, \sqrt{t})$: *i.e.,* for all intervals (a, b),

$$Pr[W(t) \in (a, b)] = \int_a^b \frac{e^{-x^2/\sigma^2 t}}{\sqrt{2\pi t}} dt$$

2. For any collection of intervals $(a_i, b_i), 1 \le i \le k$,

$$Pr[W(t_1) \in (a_1, b_1), \ldots, W(t_k) \in (a_k, b_k)] =$$

$$\int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} \prod \frac{e^{-(x_j - x_{j-1})^2/\sigma^2(t_j - t_{j-1})}}{\sqrt{2\pi(t_j - t_{j-1})}} dx_1, \ldots, dx_k$$

   Intuitively, this second condition states that Brownian motion is a Markov process: *i.e.*, the distribution of the process at time t depends only on its current value, and not at all on the evolution by which the process arrived at its current value.

3. Brownian motion is a Martingale.

9.2 Ito Processes

We now look at a generalization of Brownian motion called Ito processes. Let (W_t) be a Brownian motion. A process (Y_t) is called an *Ito process* iff there exist two adapted processes $\mu = (\mu_t)$ and $\sigma = (\sigma_t)$ *s.t.*

$$dY_t = \mu \; dt + \sigma \; dW_t$$

Note that $Y_t \sim N(\mu t, \sigma\sqrt{t})$, since $W_t \sim N(0, \sqrt{t})$.

Theorem 9.2.1 (Ito's Lemma) *If* $V : \Re \times [0, T] \to \Re$ *is a diffusion process - more specifically, a function of* Y_t *and* t*, then*

$$dV_t = \frac{\partial V}{\partial y} dY + \frac{\partial V}{\partial t} dt + \frac{1}{2} \frac{\partial^2 V}{\partial y^2} dY^2$$

PROOF.
By essentially Taylor's expansion:

$$dV_t = \frac{\partial V}{\partial y} dY + \frac{\partial V}{\partial t} dt + \frac{1}{2} \left[\frac{\partial^2 V}{\partial y^2} dY^2 + \frac{\partial^2 V}{\partial y \partial t} dY dt + \frac{\partial^2 V}{\partial t^2} dt^2 \right] + o(dt^2)$$

Note that dY^2 is of the same order as dt, since $dW \sim \sqrt{dt}$; thus, dY^2 cannot be neglected. □

9.3 Black-Scholes Model Revisited

Consider a price process where the rate of change in price is given by:

$$\frac{dS}{S} = \mu \; dt + \sigma \; dW_t \tag{9.1}$$

Equivalently,

$$\log S = \mu \; t + \sigma \; W_t + \log S_0$$

or

$$S = S_0 e^{\mu t + \sigma W_t}$$

Let $C = C(S,t)$ denote the value of a call option as a function of the price process and time. Then

$$\begin{aligned}
dC &= \{\text{by Ito's Lemma}\} \\
&= \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial t} dt + \frac{1}{2}\frac{\partial^2 C}{\partial S^2} dS^2 \\
&= \{\text{by } 9.1\} \\
&= \left(\frac{\partial C}{\partial S}\right)(\mu\ dt + \sigma\ dW_t) + \frac{\partial C}{\partial t} dt + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} dW_t^2 \\
&= \{\text{since } dW^2 \sim dt\} \\
&= \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right) dt + \sigma S \frac{\partial C}{\partial S} dW_t
\end{aligned}$$

The first term in the above expression is deterministic: *i.e.,* it is a function of time; it expresses that the processes C and S are correlated in some way. The second term is stochastic and depends on a Brownian motion.

Now consider a portfolio in which we go long on a call option and short on some stock:

$$\begin{aligned}
dC - \frac{\partial C}{\partial S} dS &= \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right) dt + \sigma S \frac{\partial C}{\partial S} dW_t - \\
&\qquad \mu S \frac{\partial C}{\partial S} dt - \sigma S \frac{\partial C}{\partial S} dW_t \\
&= \left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right) dt
\end{aligned}$$

In this final expression, we have eliminated the stochastic effects of Brownian motion. We have replicated a totally deterministic (*i.e.,* riskless) portfolio by combining a stock and a call option on that stock.

Consider a bond process (B_t) which evolves according to the following diffusion equation:

$$\frac{dB}{B} = r\ dt \tag{9.2}$$

Now since (B_t) is a totally riskless portfolio,

$$dB = dC - \frac{\partial C}{\partial S} dS$$

This implies that

$$B = C - \frac{\partial C}{\partial S} dS$$

and moreover,

$$C = B + \frac{\partial C}{\partial S} dS$$

Thus, a call option is equivalent in value to B units of bond, and $\frac{\partial C}{\partial S} dS$ units of stock.

Finally, a few simple algebraic manipulations yields the Black-Scholes equations:

$$rB\ dt = r(C - \frac{\partial C}{\partial S} dS) dt$$

and

$$rB\ dt = dB = dC - \frac{\partial C}{\partial S} dS = \left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} \right) dt$$

Combining these expressions, we derive:

$$\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} + r\ dS \frac{\partial C}{\partial S} - rC = 0$$

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