LECTURE # 5

$G = (V, E)$ Google Graph

Directed graph determined by Hyperlinks.

$P = D^{-1}(A) \longrightarrow A$ = adjacency matrix

$a_{ij} \in \{0,1\}$

$a_{ij} = 1$ iff $(v_i, v_j) \in E$

D = Diagonal Degree Matrix

$d_{ii} = \deg(v_i)$

$= |V_i|$

$= |\{v_j \mid (v_i, v_j) \in E\}|$

V_i = Neighbors of v_i

= Hyperlinked pages

$= \{v_j \mid (v_i, v_j) \in E\}$

$$p_{ij} = \frac{a_{ij}}{|V_i|} = \frac{a_{ij}}{\deg(v_i)}$$

⇒ RANDOM SURFER

when the surfer arrives at node v_i, if $V_i \neq \phi$, he chooses a hyperlink to $v_j \in V_i$ at random.

$\vec{a}$ = Dangling Node Vector $a_i \in \{0,1\}$

$a_i = 1$ iff $|V_i| = 0$ (i.e. $V_i = \phi$).

$$P' = P + a\frac{\mathbb{1}^T}{n}$$

⇒ RANDOM TELEPORTATION

When the surfer arrives at a sink node v_i ($V_i = \phi$) he teleports stochastically to a random node $v_j \in V$

α = TREMBLING HAND $0 \le \alpha \le 1$

At any node, the surfer's hand trembles with a probability $= \alpha$ and he teleports stochastically to a random node $v_j \in V$.

$$G = \alpha\frac{\mathbb{1}\mathbb{1}^T}{n} + (1-\alpha)\frac{I+P'}{2}$$

Let $W = \frac{I+P'}{2}$

= GOOGLE MATRIX

REVISE MATERIAL ON pp 34

NOTE BOOK

PAGE RANK THESIS

"A page is important, if it is pointed to by other important pages."

MARKOV THEORY.

⇒ Stationary Values of an Enormous Markov Chain

↳ Determined by $G = \alpha \frac{\mathbb{1}\mathbb{1}^T}{n} + (1-\alpha) W$

Problem with the Iterative Process.

a) $\boxed{\text{Convergence:}}$ Will this iterative process continue indefinitely?

b) $\boxed{\text{Conditioning:}}$ What properties of G guaratees convergence?

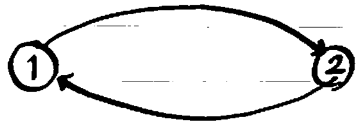
c) $\boxed{\text{Relevance:}}$ Will it converge to something relevant?

d) Uniquenes, Dependence on Initialization; ..

e) $\boxed{\text{Complexity:}}$

Start with $p_{v_i}^{(0)} = \frac{1}{n}, \ \forall v_i \in V$

CYCLE



$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = D^{-1} A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$p_0 = (1 \ \ 0) \longrightarrow p_1 = (0 \ \ 1) \longrightarrow p_2 = (1 \ \ 0) = p_0$$

Indefinite Flip-Flop

$$W = \frac{I + P}{2} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$p_0 = (1 \ \ 0) \longrightarrow p_1 = (1/2 \ \ 1/2) \longrightarrow p_2 = (1/2 \ \ 1/2) = p_1 \ \cdots$$

THEORY OF MARKOV CHAIN

G: Transition Probability Matrix

Power Method
For any starting ~~stationary~~ vector
applied to a Markov matrix G
it converges to a unique positive
vector.
~~SA~~ STATIONARY VECTOR.

G: Stochastic
&
Primitive $\rightsquigarrow$ Aperiodic
+
Irreducible.

(I) G = stochastic
A convex combination of four
stochastic matrices (row sum = 1)

$$I, \underset{D^{-1}A}{\overset{\shortparallel}{P}}, a\frac{\mathbb{1}^T}{n}, \frac{\mathbb{1}\mathbb{1}^T}{n}.$$

(I) G = Irreducible

Every page is connected to every other page (Trembling Hand Teleportation)

(II) G = Aperiodic

Self-loops ($G_{ii} > 0$) creates aperiodicity in $W = \frac{I + P'}{2}$

$\exists_k \quad G^k > 0 \qquad (\text{e.g. } G^1 > 0)$

$\Rightarrow$ G = Primitive.

$$G = (1-\alpha)\frac{P}{2} + \frac{(1-\alpha)}{2} a \frac{\mathbb{1}^T}{n} + \cancel{\alpha}(1-\alpha)\frac{I}{2} + \alpha\frac{\mathbb{1}\mathbb{1}^T}{n}$$

$$= \underbrace{(1-\alpha)\frac{I}{2}}_{\text{Diagonal Matrix}} + \underbrace{(1-\alpha)\frac{P}{2}}_{\text{Sparse Row Stochastic Matrix}} + \underbrace{\left\{\frac{(1-\alpha)}{2} a + \alpha\mathbb{1}\right\}\frac{\mathbb{1}^T}{n}}_{\text{Dense Rank-One Telepotation Matrix}}$$

NOTE BOOK

Iterative Equation: (For updating)

$$p^{(k+1)} = p^{(k)} G$$

$$p^* = p^* G$$

$p^* = \pi$ = Fixpoint.
= Page Rank Vector
= Stationary Row Vector.

Eigenvector Problem for G

$$p^* = p^* G$$
$$p^* \mathbb{1} = 1$$

$\boxed{\text{POWER METHOD}}$ $p^{(k)} \longrightarrow p^{(k+1)}$

- Sparse Update
  + Rank-One update.

MAP-REDUCE.

Matrix Free Computation.

Storage
- Sparse Matrix P
- Dangling Vector a
- Page Rank Vector $p^{(k)}$. □