SOCIAL NETWORKS

March 3 2015

Eigenvalues & Eigenvectors.

$A \in \mathbb{R}^{n\times n}$ = $n\times n$ real matrix.

λ = Eigenvalue of A,
if there exists a nonzero vector satisfying
$Ax = \lambda x \Rightarrow (A-\lambda I)x = 0$

x: Eigenvector of A associated with λ.

$Ax - \lambda x = 0$ has a nontrivial solution
$\Leftrightarrow \det(A-\lambda I) = 0$

$$\det(A-\lambda I) = (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$$
$$= \underbrace{\lambda_1\lambda_2\cdots\lambda_n}_{\det A} - \lambda\Big[\sum_i \lambda_1\cdots\lambda_{i-1}\lambda_{i+1}\cdots\lambda_n\Big]$$
$$\cdots$$
$$+ (-1)^{n-1}\lambda^{n-1}\underbrace{(\lambda_1+\lambda_2+\cdots+\lambda_n)}_{\mathrm{Tr}\,A}$$
$$+ (-1)^n\lambda^n$$

Two matrices A and B are similar if there is an invertible matrix P such that

$$A = P^{-1} B P$$

Similar matrices represent same linear transformation (but expressed in different bases).

Note $\det(A - \lambda I) = 0 \iff \exists x \;\; Ax = \lambda x$

$$\iff \exists_x \; (P^{-1}BP)x = \lambda x$$
$$\iff \exists_x \; B(Px) = \lambda Px$$
$$\iff \exists y \;\; By = \lambda y$$
$$\iff \det(B - \lambda I) = 0.$$

Two similar matrices have exactly the same eigenvalues:

SPECTRAL STRUCTURE.

A matrix A is diagonalizable
iff A is similar to a diagonal matrix
iff A has n linearly independent eigenvectors.

$\exists\, \Phi$ invertible

$$\Phi^{-1} A \Phi = \Lambda$$

A is orthogonally diagonalizable
iff there exists an orthogonal matrix Φ
{i.e. Φ is invertible and $\Phi^{-1} = \Phi^T$}
such that

$$\Phi^{-1} A \Phi = \Phi^T A \Phi = \Lambda$$

$$A = \Phi \Lambda \Phi^T$$

Columns of Φ are the eigenvectors of A.

Diagonal elements of Λ are the corresponding eigenvalues.

Let φ_i : i^{th} column of Φ

$$A \varphi_i = \Phi \Lambda \Phi^T \varphi_i = \Phi \Lambda \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\leftarrow$ ith element

$$= \Phi \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_i \varphi_i$$

Theorem: A is diagonalizable iff it has n linearly independent eigenvectors.

Real Spectral Theorem.

Let A be a real ~~spectral~~ symmetric matrix. Then

1. All its eigenvalues are real:

   $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$

2. If λ is an eigenvalue with multiplicity k then λ has k linearly independent eigenvectors.
3. A is orthogonally diagonalizable.

SENDER-RECEIVER GAMES.

2-Player Information Asymmetric Games.

Between $\begin{cases} \text{Informed Agent } (S = \text{Sender}) \\ \text{Uninformed Agent } (R = \text{Receiver}) \end{cases}$

S (Sender) R (Receiver)

$$\mathcal{D}_S \longrightarrow M \longrightarrow A_R$$

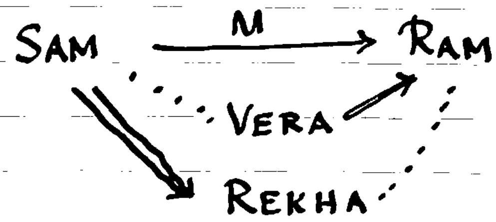
Utilities

$$\mathcal{U}_S(D, M, A) \qquad \mathcal{U}_R(D, M, A)$$

Independent (possibly, misaligned) Utility Functions.

- DECEPTION (Resulting from Information Asymmetry)
  - Spam, Malware, Bots, Ad-frauds, Identity Theft, Intrusion on Privacy, ...
  - Machine Learning, BIG DATA, Data Science, ...

⇒ Verifiers + Recommenders.



$|v|$ = # Verifiers
$|r|$ = # Recommenders

$\left\{ \begin{array}{l} |v| = |r| = 0 \\ \text{1- Click} \\ \text{"I am feeling lucky."} \end{array} \right.$

GOOGLE SIGNALING GAME.

(1) FULLY UNINFORMATIVE (PRIVATE) SENDER

User = Sender

$\mathcal{D} = \{s_0\}$; $M = A = V$ $\left\{ \begin{array}{l} \text{Keyword} \\ \downarrow \\ \text{Pages.} \end{array} \right.$

Google = Receiver.

(2) MARKOVIAN RECOMMENDER.

$G = (V, E)$ ← Directed (Page-Link) Graph.

Action at time $t = v_i$

$$V_i = \{ v_j \mid (v_i, v_j) \in E \}$$

Recommendation

If $V_i \neq \phi$ choose at time $t+1$ an element of V_i selected at random.

RANDOM SURFER MODEL (MARKOVIAN)

$$\Pr[\text{Action}_{t+1} = v_j \mid \text{Action}_1 = v_1, \dots \text{Action}_t = v_i]$$

$$= \Pr[\text{Action}_{t+1} = v_j \mid \text{Action}_t = v_i]$$

$$= \frac{a_{ij}}{\deg(v_i)} = \frac{a_{ij}}{|V_i|} = p_{ij}$$

$$P = D^{-1} A.$$

If $V_i = \phi$ choose at time $t+1$ an element of V at random.

DANGLING NODE (No recommendation)

$\vec{a}$ = Dangling Node Vector.

$$a_i = \begin{cases} 1 & \text{if } i \text{ is a dangling node} \\ 0 & \text{otherwise} \end{cases}$$

$$P' = P + a\left(\frac{\mathbb{1}^T}{n}\right)$$

3) OBLIVIOUS VERIFIER.

With probability $\alpha > 0$, ignore the recommendation and TELEPORT. (Treat $V_i = \phi$).

4) Utilities

$$U_S = U_R = \text{const.}$$

GOOGLE MATRIX.

$$G = \alpha \frac{\mathbb{1}\mathbb{1}^T}{n} + (1-\alpha)\frac{I+P'}{2}$$

(Note: We modified it a bit by introducing a self-loop $\Rightarrow$ Stay in the same page with $pr = 1/2$).

$$W \equiv \frac{I+P'}{2}$$

$p^{(t)}$ = Row vector representing the probability distribution for the random surfer occupying vertices of V at time $t \geq 0$.

$$p_{v_i}^{(t+1)} = p_{v_1}^{(t)} p_{1i} + p_{v_2}^{(t)} p_{2i} + \cdots + p_{v_n}^{(t)} p_{ni}$$

$$= \langle p^{(t)}, \underbrace{G_{,i}}_{i^{th} \text{ column of the Google Matrix.}} \rangle$$

$$G = \alpha \frac{\mathbb{1}\mathbb{1}^T}{n} + (1-\alpha) W$$

$$p^{(t+1)} = p^{(t)} G$$

$$= \alpha \frac{\mathbb{1}^T}{n} + (1-\alpha) p^{(t)} W$$

Limit Probability Distribution:

$$\pi = \alpha \frac{\mathbb{1}^T}{n} + (1-\alpha) \pi W$$

$$= \alpha \sum_{k=0}^{\infty} (1-\alpha)^k \frac{\mathbb{1}^T}{n} W^k$$

$$\pi \left(I - (1-\alpha) W \right) = \alpha \frac{\mathbb{1}^T}{n} \left[\left(I - (1-\alpha) W \right) \sum_{k=0}^{\infty} (1-\alpha)^k W^k \right]$$

$$= \alpha \frac{\mathbb{1}^T}{n} \left[I - \cancel{(1-\alpha)W} + \cancel{(1-\alpha)W} - \cdots \right]$$

$$= \alpha \frac{\mathbb{1}^T}{n}$$

$$\pi\left[I - (1-\alpha)\frac{I + P'}{2}\right] = \alpha\frac{\mathbb{1}^T}{n}$$

$$\pi\left(2\alpha I + (1-\alpha)\left(I - P'\right)\right) = 2\alpha\frac{\mathbb{1}^T}{n}$$

$$\pi\left(\beta I + \Delta\right) = \beta\frac{\mathbb{1}^T}{n}, \quad \beta = \frac{2\alpha}{1-\alpha}$$

$$\pi\left(\beta I + \mathcal{L}\right) = \beta\frac{\mathbb{1}^T}{n}$$

$\mathcal{L} = D^{1/2}\Delta D^{-1/2}$ = Laplacian.

$$\pi = \beta\frac{\mathbb{1}^T}{n}\mathcal{G}_\beta$$

G_β = Discrete Green's Function.

G_β can be computed iteratively (using MAP-REDUCE)

$$(1+\beta)\,G_\beta = I + G_\beta P'.$$

If $\mathcal{L} = \sum_{i=1}^{n-1} \lambda_i \Phi_i^T \phi_i$

$$G_\beta = \sum_{i=1}^{n-1} \frac{1}{\beta + \lambda_i}\Phi_i^T\phi_i$$