SOCIAL NETWORKS

LECTURE #9

Probability space on graphs of a given order $n \geq 1$.

1) Fix a vertex set V consisting of n distinct individuals
$$[n] = \{1, 2, \ldots, n\} = V$$
2) Fix a parameter $p \in [0,1]$.

S = Sample Space = Set of all $2^{\binom{n}{2}}$ labelled graphs with vertices V

$\mathcal{F}$: Finest σ-Algebra on S.

$\mathbb{P} : \mathcal{F} \to \mathbb{R}$ = Probability Measure

$$\mathbb{P}(G) = p^{|E(G)|} (1-p)^{\binom{n}{2} - |E(G)|}$$

$$\mathbb{P}(\phi) = 0$$

$$\mathbb{P}(S) = \sum_{|E|=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{|E|} p^{|E|} (1-p)^{\binom{n}{2} - |E|}$$

$$= \left[p + (1-p) \right]^{\binom{n}{2}} = 1.$$

$G(n,p) \equiv$
Graphs with vertex set V
s.t. two distinct vertices are
joined independently with
probability p.

→ Random Graph of
order n with edge probability p.

Erdös-Renyi Graphs.

RAMSEY THEORY

$R(3) = 6$; $R(4) = 18$;
$R(5) = 46 \pm 3$; $R(6) = 133 \pm 32$
$R(7) \in [205, 540]$; $R(8) = [282, 1870]$
...

If you have a social network of
6 people, then
1) Either it has three individuals
who know each other
2) Or it has three individuals
who do not know each other.

It has a clique of size ≥ 3
$\Rightarrow$
OR It has an independent set of size ≥ 3

$\omega(G)$ = Clique number of G
= The order of a largest clique.
$\alpha(G)$ = Stability number of G.
= The order of a largest independent set.

$n > 1$: Integer
$R(n)$ = The smallest integer m such that any graph of order m contains K_n or $\overline{K}_n$.

$$\forall_G \quad v(G) \geq R(n) \Rightarrow \omega(G) \geq n \vee \alpha(G) \geq n.$$

$R(n) \equiv$ RAMSEY NUMBER.

THEOREM 1:
For all $n \geq 3$, $R(n) \leq 2^{2n-3}$.

Ideas for Proof.
$R(r, s)$
Consider a complete graph on $R(r,s)$ vertices; color the edges of the graph red or blue;
$\Rightarrow \exists$ blue K_r or red K_s

$R(n,1) = R(1,n) = 1$

$\exists$ blue K_n or red K_1 $\exists$ blue K_1 or red K_n $\Big\}$ Trivial.

Proof by Induction:

$$R(r,s) \leq R(r-1,s) + R(r,s-1).$$

1) Consider a complete graph on $R(r-1,s) + R(r,s-1)$ vertices.
2) Color its edges red or blue.
3) Pick a vertex v from the graph

$$M = \{ w \mid (v,w) = \text{blue} \ \cancel{\text{red}} \} \qquad M \cap N = \phi$$
$$N = \{ w \mid (v,w) = \text{red} \ \cancel{\text{blue}} \}$$

4) $R(r-1,s) + R(r,s-1) = |M| + |N| + 1.$

$\Rightarrow |M| \geq R(r-1,s)$ or $|N| \geq R(r,s-1)$

5) W.LOG assume $|M| \geq R(r-1,s)$

If M has a red $K_s \Rightarrow$ done.

If M has a blue K_{r-1}

$\Rightarrow M \cup \{v\}$ has a blue K_r

$\Rightarrow$ done.

$$R(n,n) \leq 2^{2n-3}.$$

THEOREM. 2.

For each integer $n \geq 3$, $R(n) > 2^{n/2}$

Proof: $m = 2^{n/2}$ $\quad G \in G(m, 1/2)$

$$Pr\left[\omega(G) < n \wedge \alpha(G) < n\right] > 0$$

$$\Rightarrow \exists G \quad |V(G)| = m = 2^{n/2} \wedge K_n \notin G \wedge \bar{K}_n \notin G.$$

$$\Rightarrow R(n) > m = 2^{n/2}.$$

$$\mathbb{P}\left[\omega(G) \geq n\right] \leq \binom{m}{n}\left(\frac{1}{2}\right)^{\binom{n}{2}}$$

$$\leq \frac{m^n}{2^n} \, 2^{-\frac{1}{2}(n^2-n)}$$

$$\leq \frac{2^{n^2/2}}{2^n} \cdot 2^{-\frac{1}{2}(n^2-n)} = 2^{-\frac{n}{2}} < \frac{1}{2}$$

$$\mathbb{P}\left[\alpha(G) \geq n\right] \leq \binom{m}{n}\left(\frac{1}{2}\right)^{\binom{n}{2}} < \frac{1}{2}.$$

$$\mathbb{P}\left[\omega(G) < n \wedge \alpha(G) < n\right]$$

$$= 1 - \mathbb{P}\left[\omega(G) \geq n \vee \alpha(G) \geq n\right]$$

$$> 1 - \left(\frac{1}{2} + \frac{1}{2}\right) = 0. \qquad \square.$$

ON-LINE WEB GRAPH MODELS.

Models whose vertex set increases in cardinality over time.

Random Graph Process

A sequence of finite graphs G_t indexed by $(t, t \in \mathbb{N})$

(1) $G_0 \cong H$

(2) G_t = An induced subgraph of G_{t+1}

(3) $|V(G_{t+1})| = |V(G_t)| + 1$.

New vertex $= V(G_{t+1}) \setminus V(G_t)$.

$E(G_{t+1}) \setminus E(G_t) \longrightarrow$ deterministic
$\searrow$ random variable.

$N_{k,t}$ = Number of vertices of degree k at time t.

$\frac{N_{k,t}}{t} = ?$

$\mathbb{E}\left[\frac{N_{k,t}}{t}\right]$

Models.

(1) Preferential Attachment: PA.

New vertices are more likely to join existing vertices with high degree.

$$G_t \longrightarrow G_{t+1}$$

Add v_t

$$\Pr\left[(v_t, v_s) \in E_{t+1}\right] \propto \begin{cases} \dfrac{\deg_{G_t}(v_s)}{2t-1}, & 1 \leq s \leq t-1 \\ \text{or } \dfrac{1}{2t-1}, & s = t \end{cases}$$

(2) PA $\longrightarrow$ LCD PA

↳ (Linearized Chord Diagram)

(3) PA $\longrightarrow$ ACL PA

↳ (Aiello, Chung & Lu)

(4) EBD $\rightarrow$ Copying Model

(Evolution by Duplication)

(5) GROWTH - DELETION MODEL

(6) GEOMETRIC WEB GRAPH MODEL.