# Social Networks

## Lecture #7

A (discrete) random variable (r.v.) on a probability space $S$ is a function
$$X : S \to \mathbb{R}.$$

$X$ = Geometric random variable with parameter $p$.

$$p \in (0,1] \qquad S = \mathbb{N} \setminus \{0\}$$
$$\mathbb{P}(x) = p(1-p)^{x-1}$$
$$X : \mathbb{N}\setminus\{0\} \to \mathbb{R}$$
$$: x \mapsto x \qquad X(x) = x$$

Number of failed trials required to obtain the first success.

E.g. Number of cointosses until the first head appears.

$$\underbrace{TTT \cdots TH}_{x}$$

Given $x \in \mathbb{R}$, define

$$\mathbb{P}(X = x) = \sum_{\substack{s \in S \\ X(x) = s}} \mathbb{P}(\{s\})$$

$$\mathbb{P}(X \geq x) = \sum_{\substack{s \in S \\ X(x) \geq s}} \mathbb{P}(\{s\})$$

$\left.\begin{array}{l} \mathbb{P}(X > x) \\ \mathbb{P}(X \leq x) \\ \mathbb{P}(X < x) \end{array}\right\}$ Analogous

Probability mass function of $X$

$$f(x) = \mathbb{P}(X = x)$$
$$f : \mathbb{R} \to [0,1]$$

The expectation (mean, average or first ~~mom~~ moment)

of a random variable $X$

$$\mathbb{E}(X) = \sum_{s \in S} X(s)\, \mathbb{P}(\{s\})$$

If $S$ is finite, then $\mathbb{E}(X)$ is always finite.

If $X \geq 0$ (a ~~p~~ non-negative valued r.v.) the $\mathbb{E}(X) \geq 0$ = non-negative.

$X \sim$ Geometric$(p)$

Non-negative valued r.v.

$$\mathbb{E}(X) = \sum_{x \in \mathbb{N} \setminus \{0\}} x\, p(1-p)^{x-1}$$
$$= \frac{p}{(1-(1-p))^2} = \frac{1}{p}.$$

<u>Theorem</u>: Suppose that

$X$, $Y$, and $X_i$, $1 \le i \le n$

are r.v.'s defined on a probability space. Then following properties hold.

(1) LINEARITY OF EXPECTATION.

Let $c_i$, $1 \le i \le n$, be real numbers. Then

$$\mathbb{E}\left(\sum_{i=1}^{n} c_i X_i\right) = \sum_{i=1}^{n} c_i \, \mathbb{E}(X_i).$$

(2) MONOTONICITY

If $X \le Y$ (i.e. $\forall_{s \in S}\, X(s) \le Y(s)$). Then $\mathbb{E}(X) \le \mathbb{E}(Y)$.

Random Variable $X$ and $Y$ are <u>independent</u> if

$\forall_{x,y \in \mathbb{R}}$ Events $(X \le x)$ and $(Y \le y)$ are independent.

The <u>Variance</u> of a random variable $X$

$$\begin{aligned}
Var(X) &= \mathbb{E}\left((X - \mathbb{E}(X))^2\right) \\
&= \mathbb{E}\left(X^2 + \mathbb{E}(X)^2 - 2X\,\mathbb{E}(X)\right) \\
&= \mathbb{E}(X^2) + \mathbb{E}(X)^2 - 2\,\mathbb{E}(X)^2 \\
&= \mathbb{E}(X^2) - \mathbb{E}(X)^2
\end{aligned}$$

2nd moment     Square of the 1st moment.

The <u>Covariance</u> of two random variables $X$ and $Y$ defined on the same probability space is

$$\operatorname{Cov}(X,Y) = \mathbb{E}\Big(\big(X-\mathbb{E}(X)\big)\big(Y-\mathbb{E}(Y)\big)\Big).$$

INDICATOR RANDOM VARIABLE.

$$\mathbb{1}_{\text{event}} = \begin{cases} 1 & \text{if event happens} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{E}\left(\mathbb{1}_{a<x<b}\right) = \cancel{\int_{-\infty}^{\infty} \mathbb{1}_{a<x<b}\, f(x)\,dx}$$

$$= \sum_{x=-\infty}^{\infty} \mathbb{1}_{a<x<b}\, \mathbb{P}(x)$$

$$= \sum_{a<x<b} \mathbb{P}(x) = \mathbb{P}[a<x<b]$$

Let $X$ be a nonnegative r.v. Then

$$t \le x \iff 1 \le \frac{x}{t}$$

$$\mathbb{1}_{x\ge t} \le \frac{x}{t}$$

$$\mathbb{E}\left(\mathbb{1}_{x\ge t}\right) \le \mathbb{E}\left(\frac{x}{t}\right) = \frac{\mathbb{E}(X)}{t}$$

$$\mathbb{P}r[X \ge t] \le \frac{\mathbb{E}(X)}{t}.$$

THEOREM (Markov's Inequality)

Let $X \geq 0$ be a non-negative r.v. on a probability space. If $c$ is a positive real number, then

$$\mathbb{P}(X \geq c) \leq \mathbb{E}(X)/c. \quad \square$$

THEOREM (Chebyshev's Inequality)

Let $X$ be a random variable on a probability space. If $c$ is a positive real number, then

$$\begin{aligned}
&\mathbb{P}(|X - \mathbb{E}(X)| \geq c) \\
&= \mathbb{P}\left((X - \mathbb{E}(X))^2 \geq c^2\right) \\
&\leq \frac{\mathbb{E}\left((X - \mathbb{E}(X))^2\right)}{c^2} \\
&= \frac{\operatorname{Var}(X)}{c^2}.
\end{aligned}$$