# Lecture # 8

April 1 2014 

**Random Graphs:**

A general model of a social network.

→ ER (Erdős-Renyi) Random Graphs.

## TWO WAYS OF DESCRIBING RANDOM GRAPHS:

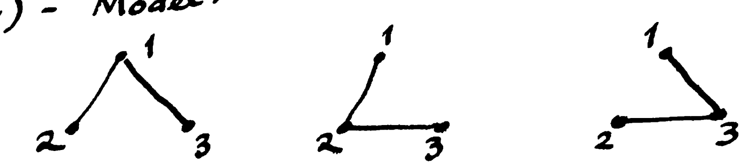
Closely related variants of ER-Random Graphs.

$\begin{cases} G(n,m) \text{ Models:} \\ G(n,p) \text{ Models:} \end{cases}$

**(A) G(n, m) Models:**

A graph $G = (V,E)$ is chosen uniformly at random from the collection of all graphs, which have $|V| = n$ nodes and $|E| = m$ edges.

$G(3,2)$ - Model.




$Pr = \{ 1/3, \quad 1/3, \quad 1/3, \}$

- Exactly three possible graphs on three vertices and two edges.
- Each is selected with equal probability $= 1/3$.

## B $G(n,p)$ Models

A graph $G=(V,E)$ is constructed by connecting every pair of nodes uniformly randomly with $Pr[e \in E] = p$.

For every pair of vertices $u, v \in V$ an edge $e = (u,v) \in E$ is included in the graph with probability $p$ independent from every other edge.

Equivalently,
All graphs with $|V| = n$ & $|E| = M$ have equal edge probability of

$$p = \frac{M}{\binom{n}{2}} = \text{Density of the graph.}$$

- At $p = 1/2$, all graph on $n$ vertices are chosen with equal probability $= p^k(1-p)^{\binom{n}{2}-k} = 2^{-\binom{n}{2}} = 2^{-\binom{n}{2}}$

- As $p$ increases from 0 to 1, the model produces denser graphs with higher probability (than sparser graphs).

- Expected Number of Edges:

$$\langle |E| \rangle = \binom{n}{2} p = \frac{n(n-1)p}{2}$$

- Expected Degree.

$$\bar{d} = \langle d \rangle = \frac{2\langle |E| \rangle}{\langle |V| \rangle} = \frac{2\binom{n}{2}p}{n} = (n-1)p.$$

There are $(n-1)$ possible other vertices, of which each can be adjacent with probability $= p$.

$$d(v) \sim \mathrm{Bin}(n-1, p)$$

$$P[d(v) = k] = \binom{n-1}{k} p^k (1-p)^{n-1-k}.$$

The degree of a vertex in a graph $G \in G(n,p)$ is distributed as a Binomial.

$$\mu[d(v)] = (n-1)p \qquad \sigma^2[d(v)] = (n-1)pq.$$

Asymptotic Analysis:

$$|V| = n \to \infty$$

Random Graphs are often studied in the asymptotic case, as $|V| = n$ (the number of vertices) tends to infinity.

$\Rightarrow$ Graphon, Graphlet.

If expected degree $\bar{d}$ is held constant (independent of $n$)

$$\bar{d} = (n-1)p = \text{const} = \lambda$$

$$p = \frac{\lambda}{n-1}$$

$$\Pr[d(v) = k] \approx \frac{(n-1)^{(k)}}{k!} p^k (1-p)^{1/p\,(n-1-k)p}$$

$$\approx \frac{[(n-1)p]^k}{k!} e^{-(n-1)p}$$

$$= \frac{\lambda^k}{k!} e^{-\lambda}$$

$$d(v) \sim \mathrm{Poisson}(\lambda)$$

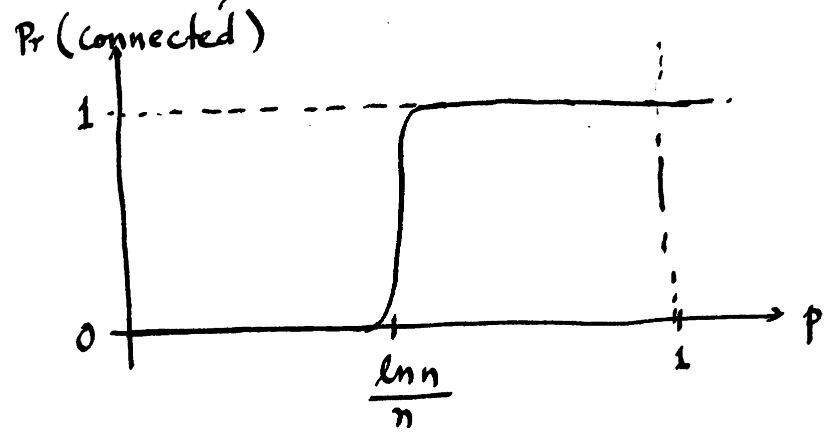
$$\left.\begin{aligned} \mu[d(v)] &= \lambda \\ \sigma^2[d(v)] &= \lambda \end{aligned}\right\} \quad \text{Poisson Approximation.}$$

## TIPPING POINT

$\begin{cases} \text{Phase Transition} \\ \text{0-1 Law.} \end{cases}$

Small $p$: If $p < \frac{(1-\varepsilon)\ln n}{n}$, then a graph in $G(n,p)$ will a.s. contain isolated vertices $\Rightarrow$ DISCONNECTED.

Large $p$: If $p > \frac{(1+\varepsilon)\ln n}{n}$, then a graph in $G(n,p)$ will a.s. be CONNECTED.




## QUESTIONS ABOUT RANDOM SOCIAL NETWORK MODELS:

1) Does the network have isolated nodes? Cycles? Giant components?

2) What are the probabilities of such events?

3) Asymptotic Analysis:
Compute Probabilities, as $n \to \infty$.

## THRESHOLD FUNCTIONS FOR CONNECTIVITY: (Erdös-Rényi 1961).

A threshold function for the connectivity of the Erdös-Rényi model $G(n,p)$ is

$$\boxed{t(n) = \frac{\ln n}{n}}$$

For a graph $G \in G(n, \lambda \frac{\ln n}{n})$

$$\Pr[G = \text{connected}] = \begin{cases} 0, & \text{if } \lambda < 1; \\ 1, & \text{if } \lambda > 1. \end{cases}$$

$$\mathbb{1}_i = \begin{cases} 1, & \text{if node } i \text{ is isolated;} \\ 0, & \text{otherwise.} \end{cases}$$

$\mathbb{1}_i \sim \text{Bernoulli}(\pi)$
$$\left\{ \begin{aligned} \pi = \Pr[\mathbb{1}_i = 1] &= (1-p)^{n-1} \\ &= (1-p)^{1/p \cdot (n-1) p} \\ &= e^{-(n-1)\lambda \frac{\ln n}{n}} \\ &\approx e^{-\lambda \ln n} = n^{-\lambda}. \end{aligned} \right.$$

$$\boxed{\mathbb{1}_i \sim \text{Bernoulli}(n^{-\lambda})} \longleftrightarrow \boxed{\pi = n^{-\lambda}}$$

~~$\neq$ Bernoulli~~

$X = \sum \mathbb{1}_i$ = Total # of isolated nodes.

~~$E[X] \neq Var[X] = n \cdot n^{-\lambda}$~~

$$E[X] = n \cdot n^{-\lambda} = n^{1-\lambda} \rightarrow \begin{cases} \infty, & \text{if } \lambda < 1; \\ 0, & \text{if } \lambda > 1. \end{cases}$$

Problems

(1) Need to show that $\Pr[X=0] = 0$, if $\lambda < 1$.

(2) Note that $\Pr[X=0] > 0$ does not necessarily imply that the graph is connected.

Assume that $\lambda < 1$.

$$\mathrm{Var}[X] = \sum_i \mathrm{var}(\mathbb{1}_i) + \sum_i \sum_{j \neq i} \mathrm{cov}(\mathbb{1}_i\ \mathbb{1}_j)$$

$$= n\ \mathrm{var}(\mathbb{1}_1) + n(n-1)\ \mathrm{cov}(\mathbb{1}_1, \mathbb{1}_2)$$

$$\begin{cases} \mathrm{var}(\mathbb{1}_1) = \pi(1-\pi) = \pi - \pi^2 \\ \mathrm{cov}(\mathbb{1}_1 \mathbb{1}_2) = E(\mathbb{1}_1 \mathbb{1}_2) - E(\mathbb{1}_1)\ E(\mathbb{1}_2) \\ \qquad = (1-p)^{2n-3} - (1-p)^{n-1}(1-p)^{n-1} \\ \qquad = \dfrac{\pi^2}{1-p} - \pi^2 \end{cases}$$

$$\mathrm{Var}[X] = n\pi - n\pi^2 + \frac{n^2\pi^2}{1-p} - n^2\pi^2$$

$$\approx n\pi + n^2\pi^2 p$$

$$= n \cdot n^{-\lambda} + n^2 n^{-2\lambda} p$$

$$\approx n^{1-\lambda} \approx E[X] \quad (\text{if } \lambda < 1.)$$

$$\mathrm{Var}[X] = \int (x - E[X])^2\, p(x)\, dx$$

$$\geq (0 - E[X])^2\, \Pr[X=0]$$

$$\Pr[X=0] \leq \cancel{\frac{E[X]^2}{\mathrm{Var}[X]} = \frac{n^{2-2\lambda}}{n^{1-\lambda}}} =$$

$$\frac{\mathrm{Var}[X]}{E[X]^2} \approx \frac{1}{E[X]} \to 0.$$

"Graph is disconnected"

$\Rightarrow \exists_{V' \subset V,\ |V'| = k,\ k \le n/2}$ Disconnect $[V', V \setminus V']$

$$\Pr[V' \text{ is not connected to } V \setminus V';\ |V| = k] = (1-p)^{k(n-k)}$$
$$= (1-p)^{1/p\ k(n-k)p}$$
$$\approx e^{-\lambda k(n-k)\ln n/n}$$

$$\Pr[\exists_{V',\ |V'|=k}\ V' \text{ is not connected to } V \setminus V']$$
$$= \binom{n}{k}(1-p)^{k(n-k)}$$
$$= \frac{n!}{k!\,(n-k)!}\, e^{-\lambda k(n-k)\ln n/n}$$

$$k = \frac{n}{2}\Big\}\ \approx \frac{\sqrt{2\pi n}\,(n/e)^n}{\cancel{2}\pi n\,(n/2e)^{n/2}(n/2e)^{n/2}}\, e^{-\lambda \frac{n^2}{4}\ln n/n}$$

$$\approx \sqrt{\frac{2}{\pi n}} \cdot 2^n \cdot n^{-\lambda n/4}$$

$$\lambda > 1\Big\}\ < \sqrt{\frac{2}{\pi n}}\ 2^{+n - n\lg n/4}$$

$$\Pr[\text{Graph is disconnected}]$$
$$\approx \sum_{k=1}^{n/2}\binom{n}{k}(1-p)^{k(n-k)}$$
$$< \frac{n}{2}\sqrt{\frac{2}{\pi n}}\ 2^{-[n\lg n/4 - n]}$$
$$= \sqrt{\frac{n}{2\pi}}\ 2^{-[n\lg n/4 - n]}$$
$$= \frac{1}{\sqrt{2\pi}}\ 2^{-[n\lg n/4 - n - \lg n/2]}$$
$$\longrightarrow 0 \quad \text{as } n \to \infty. \qquad \square$$