LECTURE # 10

April 15 2014.

GAME THEORY.

Study of strategic interactions.

$\underbrace{\text{Choices + Rational Decisions.}}_{\text{Strategic choices.}}$

- Privacy
- Trust
- Signal
- Bargaining
- Auction
- Pricing.

- Static Games
- Dynamic Games.
  - Signaling Games.
  - Bargaining Games.
- Evolutionary Games.

KEY ASSUMPTIONS (Often violated).

1) Rationality (Bounded Rationality)
2) CKR: Common Knowledge of Rationality.

Individuals (in a game/social network) act <u>rationally</u>

{ strategically select an option that optimizes their own utilities/payoffs.

a) Payoffs need not be just <u>monetary</u> { social/ Psychological/ Moral.

b) Rationality provides an <u>idealization</u> for developing a theory:

{ Bounded Rationality
Evolutionary Stable Strategies.

ORDINAL INFORMATION.

a) Set of Strategies (Options/choices): $S = \{s_1, s_2, \cdots, s_n\}$

b) Utility Function: (Real-valued)

$$u: S \to \mathbb{R}$$

i) $u(\cdot)$ represents a ranking of different options:

$$u(s_{\pi(1)}) \geq u(s_{\pi(2)}) \geq \cdots \geq u(s_{\pi(n)}).$$

c) Every strategy induces a probability distribution over consequences. (c)
(s)

$$F^{s_i}(c) \quad \text{or} \quad p_{c_j}^{s_i}$$

continuous pdf   discrete pm.

d) <u>Bernoulli Utility Function</u>: There is a utility function, called Bernoulli Utility Function, $u(c)$, which gives utility of a consequence, c.

Expected Utility Under Uncertainty:

$$u(s_i) = \begin{cases} \sum p_{c_j}^{s_i} u(c_j) \\ \quad \text{or} \\ \int u(c) f^{s_i}(c)\, dc = \int u(c)\, dF^{s_i}(c). \end{cases}$$

MULTIPLAYER SITUATION: $\begin{cases} \text{John von Neumann} \\ + \\ \text{Oskar Morgenstern.} \end{cases}$

1) Rational Decision Making (under uncertainty)
   → Proposed a set of "reasonable" axioms.
2) Expected Utility Theory
   → Under uncertainty, every choice induces a "lottery".
   (Probability Distribution over different outcomes.)

Rationality → Optimize Expected Utility:

Two actions $\begin{cases} s_a \\ \& \\ s_b \end{cases}$ ⇒ Probability Distns $\begin{cases} \mathcal{F}^{s_a}(c) \ \& \\ \mathcal{F}^{s_b}(c) \end{cases}$

⇒ Expected utilities. $\begin{cases} u(s_a) = \int u(c)\, d\mathcal{F}^{s_a}(c) \ \& \\ u(s_b) = \int u(c)\, d\mathcal{F}^{s_b}(c) \end{cases}$

Choose a over b iff $u(s_a) \geq u(s_b)$.

STRATEGIC FORM GAMES. (Defn)

A strategic form game is a triplet:

$$\langle I, (S_i)_{i\in I}, (u_i)_{i\in I} \rangle \quad \text{such that}$$

1) INDEX SET: I = Finite set of players $= \{1, 2, 3, \ldots, l\}$

2) STRATEGY SET: S_i, $i \in I$ = Set of available actions for player $i \in I$.

3) STRATEGY PROFILE: $S = \prod_i S_i$

4) ~~UT~~ UTILITY FUNCTION: $u_i : S \to \mathbb{R}$ = The payoff function of player $i \in I$.

Notations:

$S = \prod_i S_i$ = Set of all action profiles.

$s_i \in S_i$ = An action available to player i.

$S_{-i} = \prod_{j \neq i} S_j$ = Set of all strategy profiles for all players except player i.

$S \equiv S_i \times S_{-i}$

$s_{-i} \in S_{-i}$; $s_{-i} = \langle s_j \rangle_{j \neq i}$ = Vector of actions for all players excluding i.

$\langle s_i, s_{-i} \rangle$ = A strategy profile $\boxed{\text{OUTCOME}}$

Each player chooses a strategy s_i →
Generates a strategy profile → $s = \langle s_1, s_2, \ldots, s_\ell \rangle$
Obtains a utility → $u_i(s)$

How should each player make a strategic choice?

$$s^* = \langle s_1^*, s_2^*, \ldots, s_\ell^* \rangle = ?$$

Answer: $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \forall_{i \in I} \ \forall_{s_i \in S_i}$

↳ BEST RESPONSE.

1) No player can profitably deviate given the strategy of the other players. (STABILITY)

2) Each player chooses a strategy (s_i^*) expecting all other players to choose corresponding "best" strategies.

(FIXED POINT UNDER CKR)

NASH EQUILIBRIUM (PURE STRATEGY N.E.)

A pure strategy Nash Equilibrium of a strategic game:

$$\langle I, (S_i)_{i\in I}, (u_i)_{i\in I} \rangle$$

is a $\boxed{\text{STRATEGY PROFILE } s^* \in S}$ such that

$$\forall_{i \in I} \ \forall s_i \in S_i \quad u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \square$$

BoS (BATTLE OF THE SEXES) = TWO PLAYER GAME:

Two players of opposite sex $\left\{ \begin{array}{l} M = \text{Male} \\ F = \text{Female.} \end{array} \right\}$.

F = Row player M = Column player.

F \ M	Opera	Football
Opera	3,2	0,0
Football	0,0	2,3

$I = \{F, M\}$ $S_F = S_M = \{\text{Opera, Football}\}$

$$S = S_F \times S_M = \{\langle\text{opera, opera}\rangle, \langle\text{opera, football}\rangle, \langle\text{football, opera}\rangle, \langle\text{football, football}\rangle\}$$

$u_F : S \to \mathbb{R}$; $u_M : S \to \mathbb{R}$

$u_F(\langle\text{opera, opera}\rangle) = u_M(\langle\text{football, football}\rangle) = 3$

$u_F(\langle\text{football, football}\rangle) = u_M(\langle\text{opera, opera}\rangle) = 2$

$\left\{ \begin{array}{l} \text{o.w.} \\ = 0 \end{array} \right\}$

① In the matrix, in each entry

$\begin{cases} \text{First number is the payoff to } \begin{cases} \text{player 1} \\ \text{row player} \\ \text{female.} \end{cases} \\ \\ \text{Second number is the payoff to } \begin{cases} \text{player 2} \\ \text{column player} \\ \text{male.} \end{cases} \end{cases}$

② Player 1 chooses a row
$S_F \in \{\text{Opera}, \text{football}\}$
Player 2 chooses a column
$S_M \in \{\text{opera}, \text{football}\}$

Choices must be mad SIMULTANEOUSLY
-Non-cooperative-

③ The payoffs are
$u_F(S_F, S_M)$, $u_M(S_F, S_M)$

◇ GREEDY STRATEGY : $\neq$ NE

$S_F = \text{opera}$, $S_M = \text{football}$ $\Rightarrow$ payoff $= (0,0)$

S_F should deviate to football $\Rightarrow$ payoff: $0 \to 2$.
OR
S_M should deviate to opera $\Rightarrow$ payoff: $0 \to 2$.

◇ ULTRA ALTRUISTIC STRATEGY : $\neq$ NE

$S_F = \text{football}$, $S_M = \text{Opera}$ $\Rightarrow$ payoff $= (0,0)$

S_F should deviate to opera $\Rightarrow$ payoff $= 0 \to 3$
OR
S_M should deviate to football $\Rightarrow$ payoff: $0 \to 3$

◇ TWO NASH EQUILIBRIA :

$\langle$opera, opera$\rangle$ or $\langle$football, football$\rangle$

$\Downarrow$

F enjoys both the opera & M's company.
M gains utility by being in F's company.

HONESTY

AUCTION: { Used by Google, eBay, etc.

◇ SECOND PRICE AUCTION (With Complete Information)
→ Can be further relaxed.

★ Players: $I = \{1, 2, 3, \dots, n\}$
An object is to be assigned ~~to~~ to a single player $i \in I$.

★ Each player $i \in I$ has his own "private" valuation of the object:

v_i = Player i's valuation.

WLOG, assume

$$v_1 > v_2 > \dots > v_n > 0$$

★ Complete Information Version:
Assume everyone knows all the valuations:

$$V = \{v_1, v_2, \dots, v_n\}$$

→ The players simultaneously submit bids, b_i, $i \in I$

$$B = \{b_1, b_2, \dots, b_n\}$$

→ The object is assigned to the HIGHEST BIDDER (with random tie-breaking)

→ The winner pays the SECOND HIGHEST BID.

The utility function =

$$u_i(b_1, b_2, \dots, b_n) = \begin{cases} v_i - b_j & i = \text{highest bidder} \\ & j = 2^{nd} \text{ highest bidder;} \\ 0 & \text{o.w.} \end{cases}$$

HONEST BIDDING.

LEMMA: In the second price auction, HONEST BIDDING, i.e.

$$b^* = \langle v_1, v_2, \ldots, v_n \rangle \qquad \{ \text{i.e. } b_i \equiv v_i$$

is a Nash Equilibrium. { TRUTHFUL EQUILIBRIUM.

Proof:

Player 1 receives the object and pays v_2.

$$u_1(b^*) = v_1 - v_2; \quad \forall_{j \neq 1} \; u_j(b^*) = 0.$$

Player 1 has no incentive to deviate, since

a) if $b_1 > v_2$, it has no effect on $u_1(b) = v_1 - v_2$

b) if $b_1 < v_2$, it decreases his pay off to $u_1(b) = 0$.

Player $j \neq 1$ has no incentive to deviate, since

a) if $b_j > v_1$, it decrease his pay off to $u_j(b) = v_j - b_j < 0$.

b) if $b_j < v_1$, it has no effect on $u_j(b) = 0$.

No player has any incentive to deviate. □

Incomplete Information Case: (Bit more complex)

- Two additional Nas Equilibrias

$$b'^* = \langle v_1, 0, 0, \ldots, 0 \rangle$$
$$b''^* = \langle v_2, v_1, 0, \ldots 0 \rangle$$

- In general, it can be shown that HONEST BIDDING ⇒ Results in a WEAKLY DOMINANT NASH EQUILIBRIUM.