Lecture #6.

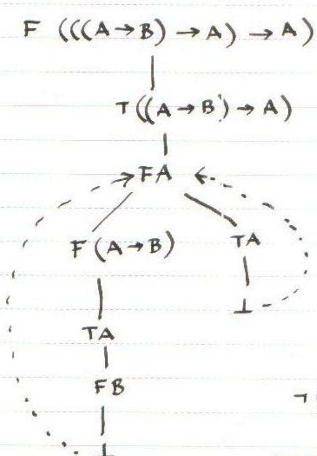
Tableu Proof for Propositional Logic.

1a TA	1b FA	2a $T(\alpha\wedge\beta)$ $T\alpha$ $T\beta$	2b $F(\alpha\wedge\beta)$ $F\alpha$ $F\beta$
3a $T(\neg\alpha)$ $F(\alpha)$	3b $F(\neg\alpha)$ $T(\alpha)$	4a $T(\alpha\vee\beta)$ $T\alpha$ $T\beta$	4b $F(\alpha\vee\beta)$ $F\alpha$ $F\beta$
5a $T(\alpha\rightarrow\beta)$ $F\alpha$ $T\beta$	5b $F(\alpha\rightarrow\beta)$ $T\alpha$ $F\beta$	6a $T(\alpha\leftrightarrow\beta)$ $T\alpha$ $F\alpha$ $T\beta$ $F\alpha$	6b $F(\alpha\leftrightarrow\beta)$ $T\alpha$ $F\alpha$ $F\beta$ $T\beta$

<u>Thm</u> (Pierce's Law)

$$(((A \to B) \to A) \to A)$$

Proof (By Tableu Method):



$$\neg(((A \to B) \to A) \to A)$$
$$\Rightarrow \bot .$$

$\boxed{\text{Proof by contradiction}}$

$\square$

<u>Atomic Tableu</u> $\equiv$ Tableu for a prime variable $\pi \in PV$.

<u>A finite Tableu</u> $\equiv$ is A binary tree labeled with signed propositions called <u>entry</u> satisfying the following inductive defn.

(i) All atomic tableus are finite tableus.
(ii) If τ is a finite tableu, P is a path on τ, E is an entry of τ occurring on P and τ' is obtained from τ by adjoining the unique atomic tableu with root entry E to τ at the end of the path P, then τ' is also a finite tableu.

If $\tau_0, \tau_1, \dots, \tau_n, \dots$ is a finite (or infinite) sequence of finite tableu such that
$\forall_{n \geq 0}$ τ_{n+1} is constructed from τ_n (by apply of (ii))
then
$\tau = \bigcup \tau_n$ is a tableu.

τ = Tableu, P = Path on τ, E = Entry on P.
(a) E = Reduced on P if all entries on one path through the atomic tableu with root E occur on P.
(b) P = Contradictory if for some proposition α, $T\alpha$ and $F\alpha$ are both entries on P.
= Finished if either contradictory or every entry is reduced.
(c) τ = Finished iff $\forall$ path P through τ P = finished.
= Contradictory iff $\forall$ path P through τ P = contradictory

A tableau proof of a proposition α is a contradictory tableau with root entry $F\alpha$.

FIRST ORDER LOGIC : SEMANTICS

Extension

7a	7b	8a	8b
$T \forall x\, \varphi(x)$	$F \forall x\, \varphi(x)$	$T \exists x\, \varphi(x)$	$F \exists x\, \varphi(x)$
$\mid$	$\mid$	$\mid$	$\mid$
$T \varphi(a)$	$F \varphi(a)$	$T \varphi(a)$	$F \varphi(a)$
(a is any parameter)	(a is a new parameter)	(a is a new parameter)	(a is any parameter)

A parameter that has not been used before

Drinking formula:

There is someone at the bar, when he drinks every one drinks.

$$F\ \exists x\,(Dx \rightarrow \forall y\, Dy) \quad [1]$$

$$F\ Da \rightarrow \forall y\, Dy$$

$$T\ Da$$

$$F\ \forall y\, Dy$$

$$F\ Db \quad (b = \text{new},\ b \neq a)$$

$$F\ Db \rightarrow \forall y\, Dy \quad (\because [1])$$

$$T\ Db$$

$$\perp$$

Formalization of First Order Logic

Semantics.

Model (Structure)

Non-trivial compared to "models" of propositional logic (determined by truth assignment).

1) SIGNATURE (Defn) Σ

A signature is a set of non-logical symbols (predicates, constants, and functions).

2) Given a signature, Σ, a model, $\mathcal{M}$, of Σ consists of the following:

a) A non-empty set, called the domain of $\mathcal{M}$, (written $dom(\mathcal{M})$).
Elements of the domain are called elements of the model $\mathcal{M}$.

b) A mapping from each constant c in Σ to an element $c^{\mathcal{M}}$ of $\mathcal{M}$.

c) A mapping from each n-ary function symbol f in Σ to $f^{\mathcal{M}} : [dom(\mathcal{M})]^n \to dom(\mathcal{M})$

d) A mapping from each n-ary predicate symbol p in Σ to $p^{\mathcal{M}} \subseteq [dom(\mathcal{M})]^n$.

$f^{\mathcal{M}}$ = n-ary function on $dom(\mathcal{M})^n \to dom(\mathcal{M})$
$p^{\mathcal{M}}$ = n-ary relation in $dom(\mathcal{M})^n$
$c^{\mathcal{M}}$ = 0-ary function on $dom(\mathcal{M})$.

Semantics.

Given a model $\mathcal{M}$, a variable assignment ρ is a function which assigns to each variable an element of $\mathcal{M}$.

Given a wff φ, we say that $\mathcal{M}$ satisfies φ with ρ, and ~~write~~ write

$$\mathcal{M} \models_\rho \varphi$$

if φ is true in the model $\mathcal{M}$ with variable assignment ρ.

More formally:

First define the extension $\bar{\rho} : T \to \mathrm{dom}(\mathcal{M})$ a function from the set T of all terms into the domain of $\mathcal{M}$.

(i) For each variable x, $\bar{\rho}(x) = \rho(x)$

(ii) For each constant c, $\bar{\rho}(c) = c^{\mathcal{M}}$

(iii) If $t_1, t_2, \cdots, t_n$ are terms and f is an n-ary function symbol, then

$$\bar{\rho}(f t_1, \cdots, t_n) = f^{\mathcal{M}}(\bar{\rho}(t_1), \cdots, \bar{\rho}(t_n))$$

{ Existence of a unique such extension $\bar{\rho}$ follows from the recursion theorem and the fact that terms are freely generated. }

Atomic Formulas.

- $\mathcal{M} \models_\rho \; = t_1 t_2$ iff $\bar{\rho}(t_1) = \bar{\rho}(t_2)$
- For an n-ary predicate symbol, P
  $\mathcal{M} \models_\rho P t_1, \dots t_n$ iff
  $\langle \bar{\rho}(t_1), \dots, \bar{\rho}(t_n) \rangle \in P^{\mathcal{M}}$

Well-Formed-Formulas

- $\mathcal{M} \models_\rho (\neg \varphi)$ iff $\mathcal{M} \not\models_\rho \varphi$
- $\mathcal{M} \models_\rho (\varphi \wedge \psi)$ iff $\mathcal{M} \models_\rho \varphi$ and $\mathcal{M} \models_\rho \psi$
- $\mathcal{M} \models_\rho \forall_x \varphi$ $\mathcal{M} \models_{(\rho)^x_d} \varphi$ for every $d \in \mathrm{dom}(\mathcal{M})$

Γ = Set of formulas (set of Σ-formulas)

$\mathcal{M} \models_\rho \Gamma \; := \; \forall_{\varphi \in \Gamma} \; \mathcal{M} \models_\rho \varphi$

$\Gamma \models \varphi \; := \; \forall_{\text{model } \mathcal{M} \text{ of } \Sigma}$
$\forall_{\text{variable assignment } \rho}$
$\mathcal{M} \models_\rho \Gamma$ implies $\mathcal{M} \models_\rho \varphi$

φ is valid iff $\models \varphi$ ($\emptyset \models \varphi$ iff $\forall_{\mathcal{M}} \forall_\rho \; \mathcal{M} \models_\rho \varphi$)

$\psi \equiv \varphi$ (ψ and φ are logically equivalent)
iff $\psi \models \varphi$ and $\varphi \models \psi$

Invariance of Truth Values:

Theorem

Suppose ρ_1 and ρ_2 are variable assignments over a model $\mathcal{M}$, which agree at all variables which occur free in the wff φ.

Then

$$\mathcal{M} \models_{\rho_1} \varphi \quad \text{iff} \quad \mathcal{M} \models_{\rho_2} \varphi.$$

Proof: By induction on wff φ.

1) φ = Atomic formula:

All variables in φ are free

$\Rightarrow$ ρ_1 and ρ_2 agree on all variables in φ.

$\Rightarrow$ $\forall$ terms t in φ $\bar{\rho}_1(t) = \bar{\rho}_2(t)$.

$\Rightarrow$ $\mathcal{M} \models_{\rho_1} \varphi$ iff $\mathcal{M} \models_{\rho_2} \varphi$.

2) $\varphi = \neg\alpha$; $\varphi = \alpha \wedge \beta$. From IH.

3) $\varphi = \cancel{\forall x\, \varphi(x)}\ \forall x \psi$.

Free Variables(φ) = Free Variables$(\psi) \setminus \{x\}$

$\Rightarrow$ $\forall_{d \in dom(\mathcal{M})}$ $(\rho_1)^x_d$ and $(\rho_2)^x_d$ agree on all variables free in ψ

$\Rightarrow$ $\mathcal{M} \models_{(\rho_1)^x_d} \psi$ iff $\mathcal{M} \models_{(\rho_2)^x_d} \psi$ $\cancel{\forall_{d \in dom(\mathcal{M})}}$ $\forall_{d \in dom(\mathcal{M})}$

$\Rightarrow$ $\mathcal{M} \models_{\rho_1} \varphi$ iff $\mathcal{M} \models_{\rho_2} \varphi$. $\square$

DEFINABILITY

$\mathcal{M}$ = A fixed model.

φ = wff; FreeVariables$(\varphi) = \{v_1, \cdots, v_k\}$

$a_1, \cdots, a_k \in \mathrm{dom}(\mathcal{M})$.

$\mathcal{M} \models \varphi [[a_1, \cdots, a_k]]$

$\mathcal{M}$ satisfies φ with some variable assignment $\rho : V \to \mathrm{dom}(\mathcal{M})$
$: v_i \mapsto a_i$

$\{\langle a_1, \cdots, a_k \rangle \mid \mathcal{M} \models \varphi [[a_1, \cdots, a_k]]\}$
$\subseteq [\mathrm{dom}(\mathcal{M})]^k$

= k-ary relation defined by φ in $\mathcal{M}$.

A k-ary relation on $\mathrm{dom}(\mathcal{M})$ is said to be definable in $\mathcal{M}$ iff there is a formula which defines it there.

$\square$