Lecture #4.

A Calculus of Natural Deduction.

- Purely Syntactic Rules
- ⇒ Formal Logic

$\vdash$ ← Derivability Relation.

Gentzen Type Calculus	Hilbert Type Calculus.
$\vdash$	$\models$

Rules are given with respect to pairs (X, α)

Formulas $\left\{ \begin{matrix} X \\ \alpha \end{matrix} \right\}$

$X \vdash \alpha$ — α is derivable (or provable) from X.

X, α ≡ Sequenz / Sequent.

Basic Rules.

(IS) $\dfrac{}{\alpha \vdash \alpha}$ (Initial Sequent)

(MR) $\dfrac{X \vdash \alpha}{X' \vdash \alpha}$, $X' \supseteq X$ (Monotonicity)

($\wedge$1) $\dfrac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta}$

($\wedge$2) $\dfrac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta}$

($\neg$1) $\dfrac{X \vdash \alpha, \neg\alpha}{X \vdash \beta}$

($\neg$2) $\dfrac{X, \alpha \vdash \beta \quad | \quad X, \neg\alpha \vdash \beta}{X \vdash \beta}$.

Derivation $\equiv$ A finite sequent

$(S_0; S_1; \dots; S_n)$

$S_n = (X, \alpha)$

$\alpha, \beta \vdash \alpha \wedge \beta$

$$\dfrac{\dfrac{\dfrac{}{\alpha \vdash \alpha}\,\text{IS}}{\alpha, \beta \vdash \alpha}\,\text{M} \qquad \dfrac{\dfrac{}{\beta \vdash \beta}\,\text{IS}}{\alpha, \beta \vdash \beta}\,\text{M}}{\alpha, \beta \vdash \alpha \wedge \beta}$$

$\dfrac{X, \neg\alpha \vdash \alpha}{X \vdash \alpha}$ $\neg$ - Elimination.

$$\dfrac{X, \neg\alpha \vdash \alpha \ (\text{H}) \qquad \dfrac{\dfrac{}{\alpha \to \alpha}\,\text{IS}}{X, \alpha \vdash \alpha}\,\text{M}}{X \vdash \alpha}\,\neg 2$$

$\dfrac{X, \neg\alpha \vdash \beta, \neg\beta}{X \vdash \alpha}$ reductio ad absurdum

$$\dfrac{\dfrac{X, \neg\alpha \vdash \beta, \neg\beta}{X, \neg\alpha \vdash \alpha}\,\neg 1}{X \vdash \alpha}\,\neg\,\text{Elim}$$

~~$X \vdash \alpha \wedge \beta$~~ → Elimination

$$\frac{X \vdash \alpha \to \beta}{X, \alpha \vdash \beta}$$

$$\dfrac{\dfrac{X, \alpha, \neg\beta \vdash \alpha, \neg\beta}{X, \alpha, \neg\beta \vdash \alpha \wedge \neg\beta}(\wedge 1) \qquad \dfrac{X \vdash \neg(\alpha \wedge \neg\beta)}{X, \alpha, \neg\beta \vdash \neg(\alpha \wedge \neg\beta)}(MR)}{\dfrac{X, \alpha, \neg\beta \vdash \beta}{X, \alpha \vdash \beta}(\neg\ Elim)} \text{red.}$$

Cut Rule

$$\frac{X \vdash \alpha \mid X, \alpha \vdash \beta}{X \vdash \beta}$$

$$\dfrac{X, \alpha \vdash \beta \qquad \dfrac{\dfrac{X \vdash \alpha}{X, \neg\alpha \vdash \alpha}(M) \qquad \dfrac{\overline{\neg\alpha \vdash \neg\alpha}\ IS}{X, \neg\alpha \vdash \neg\alpha}\ M}{X, \neg\alpha \vdash \beta}\ \text{red.}}{X \vdash \beta}\ 12$$

$$\frac{X, \alpha \vdash \beta}{X \vdash \alpha \to \beta}$$ $\to$ Introduction.

$$\frac{X, \alpha \vdash \beta}{X, \alpha \wedge \neg\beta, \alpha \vdash \beta}\ M$$

$$\frac{\dfrac{\dfrac{\overline{\alpha \wedge \neg\beta \vdash \alpha \wedge \neg\beta}\ IS}{X, \alpha \wedge \neg\beta \vdash \alpha \wedge \neg\beta}\ M}{X, \alpha \wedge \neg\beta \vdash \alpha}\ \wedge 2}{}$$

$$\frac{X, \alpha \wedge \neg\beta, \alpha \vdash \beta \qquad X, \alpha \wedge \neg\beta \vdash \alpha}{X, \alpha \wedge \neg\beta \vdash \beta}\ \text{Cut}$$

$$\frac{X, \alpha \wedge \neg\beta \vdash \beta \qquad X, \alpha \wedge \neg\beta \vdash \neg\beta}{X, \alpha \wedge \neg\beta \vdash \alpha \to \beta}\ (\text{Red})$$

$$(\neg 2)\ \frac{X, \alpha \wedge \neg\beta \vdash \alpha \to \beta \qquad \overline{X, \neg(\alpha \wedge \neg\beta)} \vdash \alpha \to \beta}{X \vdash \alpha \to \beta}$$

$$\frac{X \vdash \alpha, \alpha \to \beta}{X \vdash \beta}$$ Detachment Rule

$$\frac{\dfrac{X \vdash \alpha \to \beta}{X, \alpha \vdash \beta}\ (\to \text{elim}) \quad \Big| \quad X \vdash \alpha}{X \vdash \beta}\ \text{Cut}$$

Cor

Modus Ponen

$$\frac{\alpha, \alpha \to \beta \vdash \alpha, \alpha \to \beta\ (IS)}{\alpha, \alpha \to \beta \vdash \beta}\ (\text{Det})$$

Rule:

$$R: \frac{X_1 \vdash \alpha_1 \;|\; X_2 \vdash \alpha_2 \;|\; \cdots \;|\; X_n \vdash \alpha_n}{X \vdash \alpha}$$

Example:

Modus Ponen $$\frac{X \vdash \alpha \;|\; X \vdash \alpha \to \beta}{X \vdash \beta.}$$

[Gentzen - Style Rules].

Properties closed under R?

We wish to study a property $\mathcal{E}$

$\mathcal{E}(X_1, \alpha_1)$; $\mathcal{E}(X_2, \alpha_2)$; $\cdots$ $\mathcal{E}(X_n, \alpha_n)$ implies $\mathcal{E}(X, \alpha)$.

Ex. Soundness Property $\mathcal{E}(X, \alpha) \equiv X \models \alpha$.

We wish to show that this property is closed under the basic rules of $\vdash$.

That is if we prove that

$$X \vdash \alpha$$

then in fact $X \models \alpha$

That is $\forall_\omega \; \omega \models X \to \omega \models \alpha$

"All models of X are also models of α."

Proof theoretic "truth" $\subseteq$ Model theoretic "truth"

$\mathcal{E}: X \models \alpha$, this property applies to all provable sequents.

Thus, the relation $\vdash$ is (semantically) sound.

PRINCIPLE OF RULE INDUCTION.

Let $\mathcal{E} \left(\subseteq \mathcal{P}\mathcal{F} \to \mathcal{F} \atop : X \mapsto \alpha \right)$ be a property closed under all basic rules of $\vdash$.

Then $X \vdash \alpha$ implies $\mathcal{E}(X, \alpha)$

Proof: By induction on the length of a derivation $S = (X, \alpha)$

SOUNDNESS

" $\vdash \subseteq \models$ "
More explicitly,

$$X \vdash \alpha \Rightarrow X \models \alpha \quad \forall_{X, \alpha}.$$

ONLY SHOW FOR X = Finite.

(IS) $\alpha \vdash \alpha \Rightarrow \alpha \models \alpha \quad \forall_\omega \; \omega \models \alpha \Rightarrow \omega \models \alpha$

(MR) $\dfrac{X \vdash \alpha}{X' \vdash \alpha} (X' \supseteq X) \Rightarrow X \models \alpha$ implies $X' \models \alpha$

$\forall_\omega \; \omega \models X' \Rightarrow \omega \models X \Rightarrow \omega \models \alpha$

(∧1) $\dfrac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta}$ ⇒ $X \models \alpha, \beta$ implies $X \models \alpha \wedge \beta$

$\forall_\omega \ \omega \models X \Rightarrow \omega \models \alpha$ and $\omega \models \beta$

$\Rightarrow \omega \models \alpha \wedge \beta$.

(∧2) $\dfrac{X \vdash \alpha \wedge \beta}{X \vdash \alpha, \beta}$ ⇒ $X \models \alpha \wedge \beta$ implies $X \models \alpha, \beta$

$\forall_\omega \ \omega \models X \Rightarrow \omega \models \alpha \wedge \beta$

$\Rightarrow \omega \models \alpha$ and $\omega \models \beta$

(¬1) $\dfrac{X \vdash \alpha, \neg\alpha}{X \vdash \beta}$ ⇒ $X \models \alpha, \neg\alpha$ implies $X \models \beta$

$\forall_\omega \ \omega \models X \Rightarrow \omega \models \alpha, \ \omega \models \neg\alpha$

$\Rightarrow \forall_\omega \ \omega \not\models X$

Vacuously, $\forall_\omega \ \omega \models X$ implies $\omega \models \beta$.

(¬2) $\dfrac{X, \alpha \vdash \beta \mid X, \neg\alpha \vdash \beta}{X \vdash \beta}$ ⇒

~~X~~ $X, \alpha \models \beta$ and $X, \neg\alpha \models \beta$ implies $X \models \beta$

$\forall_\omega \ \omega \models X, \alpha$ and $\omega \models X, \neg\alpha$

$\Rightarrow \omega \models X \Rightarrow \omega \models \beta$

FINITENESS THEOREM for $\vdash$

Thm

If $X \vdash \alpha$ then there is a finite subset $X_0 \subseteq X$ with $X_0 \vdash \alpha$.

Proof: By induction.

Let $\mathcal{E}(X, \alpha)$ be the property

$$\boxed{\exists\, X_0 \subseteq X,\ X_0 = \text{finite} \quad X_0 \vdash \alpha}$$

Show that $\mathcal{E}$ is closed under the basic rules.

IS $\quad \dfrac{}{\alpha \vdash \alpha} \quad \mathcal{E}(X, \alpha) \quad X = \{\alpha\}$, $X_0 = X$.

MR $\quad \dfrac{X \vdash \alpha}{X' \vdash \alpha}\ (X' \supseteq X) \quad \left.\begin{array}{l}\mathcal{E}(X, \alpha)\\ X_0 \subseteq X\end{array}\right\}$ IH.

$\Rightarrow \mathcal{E}(X', \alpha)$

$\wedge 1 \quad \dfrac{X \vdash \alpha, \beta}{X \vdash \alpha \wedge \beta} \quad \begin{array}{ll}\mathcal{E}(X, \alpha) & \mathcal{E}(X, \beta)\\ X_1 \subseteq X & X_2 \subseteq X\end{array}$

$X_0 = X_1 \cup X_2 = \text{finite}$

$$\dfrac{X_0 \vdash \alpha, \beta}{X_0 \vdash \alpha \wedge \beta}$$

$\hookrightarrow \mathcal{E}(X, \alpha \wedge \beta)$

⋮

SATISFIABILITY:

Two cases:

HORN CLAUSES — HORN SAT.

KROM CLAUSES — 2 SAT

A Horn clause is an OR of literals in which all or nearly all of the literals are complemented [at most one of its literals is pure].

$$\bar{x} \vee \bar{y} \equiv x \wedge y \Rightarrow \bot$$

$$x \equiv T \Rightarrow x$$

$$w \vee \bar{y} \vee \bar{z} \equiv y \wedge z \Rightarrow w$$

$$\bar{u} \vee \bar{v} \vee \bar{w} \vee \bar{x} \vee \bar{y} \vee z$$

$$\equiv u \wedge v \wedge w \wedge x \wedge y \Rightarrow z$$

A Krom clause is an OR of exactly 2 literals.

$$x \vee x \equiv \{T \Rightarrow x;\ \bar{x} \Rightarrow \bot\}$$

$$\bar{x} \vee \bar{x} \equiv \{T \Rightarrow \bar{x};\ x \Rightarrow \bot\}$$

$$x \vee y \equiv \{\bar{x} \Rightarrow y;\ \bar{y} \Rightarrow x\}$$

$$x \vee \bar{y} \equiv \{\bar{x} \Rightarrow \bar{y};\ y \Rightarrow x\}$$

$$\bar{x} \vee \bar{y} \equiv \{x \Rightarrow \bar{y};\ y \Rightarrow \bar{x}\}$$

HORNSAT Algorithm.

(1) Assign all variables **false**.

Thus, initially all clauses of the following forms will be satisfied

$$x \wedge y \Rightarrow \bot$$

$$u \wedge v \wedge w \wedge x \wedge y \Rightarrow z.$$

But not

$$\top \Rightarrow x$$

$$\Downarrow$$

(2) FLIP: change x's truth assignment to **true**

(3) Reevaluate all clauses.

if ~~$x \wedge y \Rightarrow \bot$ is SAT~~
$u \wedge v \wedge w \wedge x \wedge y$ evals **true**
FLIP z's truth value to **true**
and continue.

if $x \wedge y$ evals to **true**
$x \wedge y \Rightarrow \bot$ is not satisfiable

Complexity $O(m * n)$ Every v'ble can change only from false $\rightarrow$ true.
Every clause is reevaluated after a change in assignment

HORNSAT $\in$ P.

Note: $O(m+n)$ with clever data str.

2-SAT

Create a directed graph $G = (V, E)$

$V: \{\top, \bot\} \cup \{x, \bar{x} \mid \forall x \in PV\}$

$(u, v) \in E$ iff $u \Rightarrow v$ is a clause

2-CNF ≡ Satisfiable

iff

No strong component of the digraph $G(V,E)$ contains both a variable and its complement.

Complexity $O(m+n)$

$2SAT \in P$.