$\boxed{\text{Lecture \#3}}$ (1)

Logical / Semantic Equivalence

$\Rrightarrow$ Equivalence Relation on $\mathcal{F}$

(Reflexivity) $\alpha \equiv \alpha \quad \forall \alpha$

(Symmetry) $\alpha \equiv \beta \Rightarrow \beta \equiv \alpha \quad \forall \alpha, \beta$

(Transitivity) $\alpha \equiv \beta \wedge \beta \equiv \gamma \Rightarrow \alpha \equiv \gamma \quad \forall \alpha \beta, \gamma$

$\Rrightarrow$ Congruence Relation on $\mathcal{F}$

$\forall \alpha, \alpha', \beta, \beta'$

$$\alpha \equiv \alpha' \wedge \beta \equiv \beta'$$

$$\Rightarrow \quad \alpha \circ \beta \equiv \alpha' \circ \beta' \qquad \circ \in \{\wedge, \vee\}$$

$$\neg \alpha \equiv \neg \alpha'$$

Replacement Theorem:

$$\alpha \equiv \alpha' \Rightarrow \varphi \equiv \varphi[\alpha/\alpha']$$

$\hookrightarrow$ obtained from φ by replacing one or several of the possible occurrences of the subformula α in φ by α'.

Every Boolean function can be represented by a Boolean formula.

NORMAL FORMS

1) Literals: Defn: Prime formulas and negations of prime formulas are called literals.

2) Disjunctive Normal Form (DNF)
Defn: A disjunction
$$\alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n$$
where each α_i is a conjunction of literals, is called a Disjunctive Normal Form (DNF).

3) Conjunctive Normal Form (CNF)
Defn A conjunction
$$\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n$$
where each β_i is a disjunction of literals, is called a Conjunctive Normal Form (CNF).

THEOREM [Constructive: Proof by Induction].
Every Boolean function f with

$$f \in B_n \ (n > 0)$$

is representable by a DNF,
namely by

$$\alpha_f := \bigvee_{f\vec{x}=1} p_1^{x_1} \wedge \cdots \wedge p_n^{x_n}$$

[At the same time f is representable by ~~to~~ a CNF, namely by

$$\beta_f := \bigwedge_{f\vec{x}=0} p_1^{\neg x_1} \vee \cdots \vee p_n^{\neg x_n} \ ]$$

Notation: $p_i^1 := p_i \qquad p_i^0 := \neg p_i$

$w(p_1^{x_1} \wedge p_2^{x_2}) = 1$ iff $wp_1 = x_1$ & $wp_2 = x_2$

$w(p_1^{\neg x_1} \vee p_2^{\neg x_2}) = 0$ iff $wp_1 = \neg x_1$ & $wp_2 = \neg x_2$

Proof:

By defn of α_f

$$\omega(\alpha_f) = 1 \Leftrightarrow \exists \vec{x} \quad f\vec{x} = 1$$
$$\wedge\ \omega(p_1^{x_1} \wedge p_2^{x_2} \dots \wedge p_n^{x_n}) = 1$$

$$\Leftrightarrow \exists \vec{x} \quad f\vec{x} = 1 \ \& \ \omega\vec{p} = \vec{x}$$
$$\Leftrightarrow f\omega\vec{p} = 1$$

$$\omega\alpha_f = 1 \quad \text{iff} \quad f\omega\vec{p} = 1$$

Since there are only two values

$$\omega\alpha_f = 0 \quad \text{iff} \quad f\omega\vec{p} = 0$$

$$\therefore \quad \omega\alpha_f \equiv f\omega\vec{p} \quad \forall\omega$$

The rest follows from de Morgan's Law. □

<u>Corollary</u>:

Each $\varphi \in \mathcal{F}$ is equivalent to a DNF or a CNF.

FUNCTIONAL COMPLETENESS:

A logical signature is called <u>functionally complete</u>, if every Boolean formula is representable in this signature.

<u>Examples</u>

(1) $\{\neg, \wedge, \vee\} \longrightarrow$ CNF or DNF

(2) $\{\neg, \wedge\} \longrightarrow$ de Morgan's Rule

(3) $\{\neg, \vee\}$ "

(4) $\{\rightarrow, \bot\}$

$$\neg p \equiv p \rightarrow \bot$$
$$p \vee q \equiv \neg p \rightarrow q$$
$$\equiv (p \rightarrow \bot) \rightarrow q$$

(5) $\{\downarrow\}$ NOR

(6) $\{\uparrow\}$ NAND

$$\neg p \equiv p \downarrow p \equiv p \uparrow p$$
$$p \wedge q \equiv \neg p \downarrow \neg q \equiv (p \downarrow p) \downarrow (q \downarrow q)$$
$$p \vee q \equiv \neg p \uparrow \neg q \equiv (p \uparrow p) \uparrow (q \uparrow q)$$

TAUTOLOGIES & LOGICAL CONSEQUENCES.

$\omega \models \alpha$ (ω satisfies α)

$\Leftrightarrow \omega\alpha = 1$

$\models$ ~~Sta~~ "Satisfiability Relation"

X = Set of formulas.

$\omega \models X \Leftrightarrow \forall_{\alpha \in X}\ \omega \models \alpha$

$\Leftrightarrow \forall_{\alpha \in X}\ \omega\alpha = 1.$

ω is a (propositional) model of α.

A given α (resp. X) is satisfiable if $\exists_\omega$ with

$\omega \models \alpha$ (or resp. $\omega \models X$).

$p \in PV$

$$\omega \models p \iff \omega p = 1;$$
$$\omega \models \neg\alpha \iff \omega \not\models \alpha;$$
$$\omega \models \alpha \wedge \beta \iff \omega \models \alpha \text{ and } \omega \models \beta$$
$$\omega \models \alpha \vee \beta \iff \omega \models \alpha \text{ or } \omega \models \beta$$

One may <u>define the satisfiability</u> relation $\omega \models \alpha$ for a given

$$\omega : PV \to \{0, 1\}$$

<u>inductively</u> on α

$\omega : PV \to \{0,1\}$ ← uniquely determined → By all $p \in PV$ for which $\omega \models p$ is valid.

<u>SAT.</u>

<u>Given</u> α (or X)

<u>Find</u> a map $\omega : PV \to \{0,1\}$ s.t. $\omega \models \alpha$ (or $\omega \models X$).

Defn

A wff α is called logically valid (or a Tautology) $\models \alpha$,

whenever $\omega \models \alpha$, for all valuations ω.

$\boxed{\alpha \equiv \top}$ Top

A wff α is called a contradiction

$\not\models \alpha$

whenever $\omega \not\models \alpha$, for all valuations, ω.

$\boxed{\alpha \equiv \bot}$

Example

$\models \alpha \vee \neg\alpha$ $\left\{\begin{array}{l}\text{Tertium non datur}\\ \text{Law of Excluded Middle.}\end{array}\right.$

$\left.\begin{array}{l}\not\models \alpha \wedge \neg\alpha\\ \not\models \alpha \leftrightarrow \neg\alpha\end{array}\right\}$ Contradictions.

$\models \alpha \to \alpha$ (Self-implication)

$\models (p \to q) \to (q \to r) \to (p \to r)$
(Chain rule)

$\models (p \to q \to r) \to (q \to p \to r)$
(Exchange of premises)

(*)

$\models (p \to q \to r) \to (p \to q) \to (p \to r)$ (Frege's Formula)

$\models ((p \to q) \to p) \to p$ (Peirce's Formula)

$\models p \to q \to p$ (Premise charge).

All tautologies in $\to$ alone are derivable from (*) — Last 3 formulas.

SAT : Is a formula satisfiable?
NP-Complete.

TAUT : Is a formula a tautology?
co-NP-complete.

EQUIV: Are two formulas equivalent?
$\alpha \equiv \beta$?
co-NP-complete.

$\forall \omega \quad \omega \models \alpha$ (Tautology)
$\forall \omega \quad \omega \models (\alpha \leftrightarrow \beta)$ (Equivalence)
$\exists \omega \quad \omega \models \alpha$ (Satisfiability)

Defn: α is a logical consequence of X, written

$$X \models \alpha$$

if $\omega \models \alpha$ $\forall$ model ω of X.

That is,

$$\forall_{\text{valuation } \omega} \quad \omega \models X \Rightarrow \omega \models \alpha.$$

Note

α is a tautology if

$$\phi \models \alpha.$$

Examples.

(a) $\alpha, \beta \models \alpha \wedge \beta$; $\alpha \wedge \beta \models \alpha, \beta$

(b) $\alpha, \alpha \rightarrow \beta \models \beta$ ← $\boxed{\text{Modus Ponens}}$

(c) $X \models \perp \Rightarrow X \models \alpha$ for all α.

(d) $X, \alpha \models \beta$ & $X, \neg\alpha \models \beta$

$\Rightarrow X \models \beta$.

Properties of $\models$ (satisfaction relation)

(R) Reflexivity $\alpha \in X$ $\quad X \models \alpha$ (In particular $\alpha \models \alpha$)

(M) Monotonicity $X \models \alpha$ & $X \subseteq X'$

$\Rightarrow X' \models \alpha$

(T) Transitivity $X \models Y$ & $Y \models \alpha$

$\Rightarrow X \models \alpha$

FINITARY

(F) $X \models \alpha \Rightarrow X_0 \models \alpha$
For some finite subset $X_0 \subseteq X$

DEDUCTION THEOREM.

(D) $X, \alpha \models \beta \Rightarrow X \models \alpha \to \beta$

$X, \alpha \models \beta$ & ω = Model for X, i.e $\omega \models X$

(i) $\omega \models \alpha \Rightarrow \omega \models \beta \Rightarrow \omega \models \alpha \to \beta$

(ii) $\omega \not\models \alpha \Rightarrow \omega \models \alpha \to \beta \quad (\because \omega \models \neg\alpha)$

$\Rightarrow \forall\omega \ \ \omega \models X \Rightarrow \omega \models \alpha \to \beta$

Hence $X \models \alpha \to \beta$

Iterated Applns of (D)

$\alpha_1, \alpha_2, \cdots \alpha_n \models \beta$

$\Leftrightarrow \alpha_1 \to \alpha_2 \to \cdots \to \alpha_n \to \beta$

$\Leftrightarrow (\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n) \to \beta$

Example

$p, q \models p$

$\Leftrightarrow p \models q \to p$

$\Leftrightarrow \models p \to q \to p$