Recall "Application" is to show that

UGC $\Rightarrow$ GW is optimal. We need

- Negative correlations $\rho \in (-1,0)$
- "low degree influences" $\mathrm{Inf}_i^{\leq k}(f)$.

$\boxed{\begin{array}{l}\text{Majority is stablest (w/ +vely correlated noise).}\\ \text{Majority is unstablest (w/ -vely correlated noise).}\end{array}}$

— × —

Low degree influences

Def $\mathrm{Inf}_i^{\leq k}(f) = \sum_{\substack{i \in S \\ |S| \leq k}} \hat{f}(S)^2.$

**Theorem ②** Fix $\rho \in (0,1)$. For every $\varepsilon > 0$, there is $\delta = \delta(\rho, \varepsilon)$ s.t. for $\boxed{k = k(\rho, \varepsilon)}$

- $f : \{-1,1\}^n \to [-1,1]$, $\mathbb{E}[f] = 0$,
- $\boxed{\mathrm{Inf}_i^{\leq k}(f) \leq \delta}$ $\forall\, 1 \leq i \leq n$, we have

$$\mathrm{Stab}_\rho(f) \leq 1 - \frac{1}{\pi} \cos^{-1}\rho + \varepsilon.$$

**Proof** construct $g : \{-1,1\}^n \to [-1,1]$ s.t.

- $\text{Stab}_\rho(f) \leq \text{Stab}_\rho(g) + \varepsilon$.
- $\text{Inf}_i(g) \leq \text{Inf}_i^{\leq k}(f) + \delta \leq 2\delta$.

- one could try $g = \sum_{|S| \leq k} \hat{f}(S) \chi_S$ ("hard truncation"), but then $g$ won't necessarily be in $[-1,1]$.

- Insted "soft truncation" $\equiv$ "smoothening" $\equiv$ Beckner operator. $\equiv$ "supress high degree Fourier coefficients."

**Def** For $\gamma > 0$, $f : \{-1,1\}^n \to [-1,1]$,

$$T_{1-\gamma}(f) = \sum_S \hat{f}(S) (1-\gamma)^{|S|} \cdot \chi_S .$$

**Claim** Equiv. $T_{1-\gamma}(f)(x) = \underset{y \sim_{1-\gamma} x}{\mathbb{E}} [f(y)]$.

Consider the R.H.S. as a function of $x$.

$$\mathop{\mathbb{E}}_{y \sim_{1-\gamma} x} \left[ f(y) \right] = \mathop{\mathbb{E}}_{y \sim_{1-\gamma} x} \left[ \sum_S \hat{f}(S) \chi_S(y) \right]$$

$\mu$ = noise

$\mathbb{E}[\mu_i] = 1-\gamma$

$$= \sum_S \hat{f}(S) \mathop{\mathbb{E}}_{\mu} \left[ \chi_S(x\mu) \right]$$

$$= \sum_S \hat{f}(S) \chi_S(x) \mathop{\mathbb{E}}_{\mu} \left[ \chi_S(\mu) \right]$$

$$= \sum_S \hat{f}(S) (1-\gamma)^{|S|} \chi_S(x).$$

$$= T_{1-\gamma}(f)(x).$$

$\blacksquare$.

<u>Fact</u> If $f \in [-1,1]$ then $T_{1-\gamma}(f) \in [-1,1]$.

## <u>Proof of Theorem 2</u>

Let $g = T_{1-\gamma}(f)$ for small enough $\gamma > 0$ and apply Theorem 1 to $g$.

<u>Claim</u>. $\mathbb{E}[g] = 0$. $\because \mathbb{E}[g] = \hat{g}(\phi) = \hat{f}(\phi) = 0.$

<u>Claim</u>. $\mathrm{Inf}_i(g) \leq 2\delta \quad \forall i.$

<u>Proof</u>

$$\mathrm{Inf}_i(g) = \sum_{i\in S} \hat{g}(S)^2$$

$$= \sum_{i\in S,\ |S|\leq k} \hat{g}(S)^2 + \sum_{i\in S,\ |S|>k} \hat{g}(S)^2$$

$$\leq \sum_{i\in S,\ |S|\leq k} \hat{f}(S)^2 + \sum_{i\in S,\ |S|>k} \hat{f}(S)^2 (1-\gamma)^{2k}$$

$$\leq \mathrm{Inf}_i^{\leq k}(f) + (1-\gamma)^{2k}$$

$$\leq \delta + \delta \qquad \text{for}$$

$$= 2\delta. \qquad k \geq O\left(\frac{1}{\gamma}\log\left(\frac{1}{\delta}\right)\right).$$

∎

<u>Claim</u> $\mathrm{Stab}_\rho(g) \leq 1 - \frac{1}{\pi}\cos^{-1}\rho + \varepsilon.$

<u>Proof</u> Applying Theorem ① to $g$.

We finish by showing $\mathrm{Stab}_\rho(f) \leq \mathrm{Stab}_\rho(g) + \varepsilon.$

$$
\begin{aligned}
\mathrm{Stab}_\rho(f) &= \frac{1}{2} + \frac{1}{2}\sum_S \hat{f}(S)^2 \rho^{|S|} \\
&= \frac{1}{2} + \frac{1}{2}\sum_S \hat{f}(S)^2 (1-\gamma)^{2|S|} \cdot \rho^{|S|} + \\
&\qquad \underline{\frac{1}{2}\sum_S \hat{f}(S)^2 \left(1-(1-\gamma)^{2|S|}\right) \rho^{|S|}} \\
&= \mathrm{Stab}_\rho(g) + \underline{\varepsilon}
\end{aligned}
$$

provided $\gamma$ is sufficiently small.

$\forall S$, either $\rho^{|S|} \leq \varepsilon$

or $$1-(1-\gamma)^{2|S|} \leq 1-(1-2|S|\gamma) = 2|S|\gamma \leq \varepsilon.$$

$\blacksquare$.

## Negative correlations

- So far $\rho \in (0,1)$, and we focused on maximizing $\text{Stab}_\rho(f)$, subject to $\mathbb{E}[f]=0$.
- Now we let $\rho \in (-1,0)$, and insted focus on minimizing $\text{Stab}_\rho(f)$, without any restriction on $\mathbb{E}[f]$.

What we really need towards the "application" is

$\boxed{\text{Theorem (3)}}$ Fix $\rho \in (-1,0)$. For every $\varepsilon > 0$, there is $\delta = \delta(\rho,\varepsilon)$, $k = k(\rho,\varepsilon)$ s.t. for

- $f : \{-1,1\}^n \to [-1,1]$,
- $\text{Inf}_i^{\leq k}(f) \leq \delta \quad \forall i.$ then

$$\text{Stab}_\rho(f) \geq 1 - \frac{1}{\pi}\cos^{-1}\rho - \varepsilon.$$

Proof We let $\rho^* = -\rho > 0$

$$f^* = \frac{f(x) - f(-x)}{2}$$

and apply Theorem (2) to $f^*, \rho^*$ !.

Claim $f^* = \sum_{|S| \text{ odd}} \hat{f}(S)\, \chi_S$.

Proof
$$f^*(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$$
$$= \sum_S \hat{f}(S)\left(\chi_S(x) - \chi_S(-x)\right) \cdot \frac{1}{2}$$
$$= \sum_{|S| \text{ odd}} \hat{f}(S)\, \chi_S(x) \quad \blacksquare.$$

Claim $\mathrm{Inf}_i^{\leq k}(f^*) \leq \mathrm{Inf}_i^{\leq k}(f) \leq \delta$.

Proof Obvious since $f^*$ only keeps odd degree part of $f$ and

$$\mathrm{Inf}_i^{\leq k}(f) = \sum_{\substack{i \in S \\ |S| \leq k}} \hat{f}(S)^2 \quad \blacksquare$$

Claim $f^*$ is also $[-1,1]$ valued.

Proof Since $f^*(x) = \frac{f(x) - f(-x)}{2}$. $\blacksquare$.

Applying Theorem ② to $f^*, \rho^*$ we have

$$\begin{aligned} \mathrm{Stab}_{\rho^*}(f^*) &\leq 1 - \frac{1}{\pi} \cos^{-1}\rho^* + \varepsilon \\ &= 1 - \frac{1}{\pi}(\pi - \cos^{-1}\rho) + \varepsilon \qquad \because \rho^* = -\rho \\ &= \frac{1}{\pi}\cos^{-1}\rho + \varepsilon. \qquad \text{---} \; \#4 \end{aligned}$$

We now finish the proof as below.

$$\begin{aligned} \mathrm{Stab}_\rho(f) &= \frac{1}{2} + \frac{1}{2}\sum_S \hat{f}(S)^2 \rho^{|S|} \\ &\geq \frac{1}{2} + \frac{1}{2}\sum_{|S| \text{ odd}} \hat{f^*}(S)^2 \rho^{|S|} \qquad \because f^* \text{ only keeps odd coefficients} \\ &= \frac{1}{2} + \frac{1}{2}\sum_{|S| \text{ odd}} \hat{f^*}(S)^2 (-\rho^*)^{|S|} \\ &= \frac{1}{2} - \frac{1}{2}\sum_S \hat{f^*}(S)^2 \rho^{*|S|} \end{aligned}$$

$$= 1 - \left( \frac{1}{2} + \frac{1}{2} \sum_S \hat{f^*}(S)^2 \rho^{*|S|} \right)$$

$$= 1 - \mathrm{Stab}_{\rho^*}(f^*)$$

$$\geq 1 - \frac{1}{\pi} \cos^{-1} \rho - \varepsilon \quad \text{by } (\#4).$$

▨.

To recall,

We need Theorem (3) towards showing that UGC $\Rightarrow$ GW is optimal.