Linearity Testing

Multiplicative not^n.	Additive not^n.
$x \cdot y$	$a \oplus b$ (add over $\mathbb{F}_2$)
$+1$	0
-1	1
$\chi_\alpha(x_1, \dots, x_n) = \prod_{i \in \alpha} x_i$	$\bigoplus_{i \in \alpha} a_i$ (linear over $\mathbb{F}_2$)

Recall $\chi_\alpha : \{-1,1\}^n \to \{-1,1\}$. $\chi_\alpha(x) = \prod_{i \in \alpha} x_i$;

is a linear function.

Note

1. $\chi_\alpha(x)\, \chi_\alpha(y) = \chi_\alpha(xy)$ where

   $(xy)_i = x_i y_i$ for $i \in \{1, 2, \dots, n\}$.

   This is precisely what a linear function is (in additive not^n,

- $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is linear iff
  - $f(a \oplus b) = f(a) \oplus f(b) \ \forall a, b$
  - or - $f(a) = \bigoplus_{i \in \alpha} a_i$ for some $\alpha \subseteq [n]$

2. $\chi_\alpha(x)\, \chi_\beta(x) = \chi_{\alpha \Delta \beta}(x)$.

proof $\chi_\alpha(x)\, \chi_\beta(x) = \prod_{i \in \alpha} x_i \prod_{j \in \beta} x_j$

$$= \prod_{i \in \alpha \Delta \beta} x_i = \chi_{\alpha \Delta \beta}(x).$$

Linearity Testing Problem

- Given $A : \{-1,1\}^n \to \{-1,1\}$, test whether A is linear.
- 3 queries.

- (Comp.). If A is linear, i.e. if $A = \chi_\alpha$, then $\Pr[\text{Test accepts}] = 1$.
- (Sound). If A is "far" from being liner, then $\Pr[\text{Test accepts}]$ is "far" from 1.

<u>Recall</u>
- $\Delta(A, B) = \Pr_x [A(x) \neq B(x)]$.
- $\Delta(A, \chi_\alpha) = \frac{1}{2} - \frac{1}{2}\hat{A}_\alpha$.

<u>Theorem</u> <u>[BLR]</u> <u>(Linearity Testing)</u>
- 3 query test, perfect completeness.
- (Sound). If $\Delta(A, \chi_\alpha) \geq \delta \quad \forall \alpha \subseteq [n]$ then $\Pr[\text{Test Accepts}] \leq 1 - \delta$.

<u>Test</u>
- Pick $x, y \in \{-1, 1\}^n$ at random.
- Accept iff $A(x \cdot y) = A(x) \cdot A(y)$.

(Soundness)

$$\Pr[\text{Accept}] = \mathop{\mathbb{E}}_{x,y}\left[\frac{1 + A(x)A(y)\cdot A(xy)}{2}\right]$$

$$= \frac{1}{2} + \frac{1}{2}\mathop{\mathbb{E}}_{x,y}\left[A(x)A(y)A(xy)\right].$$

Substitute,

$$A(x) = \sum_{\alpha} \hat{A}_{\alpha} \chi_{\alpha}(x) \qquad \alpha \subseteq [n]$$

$$A(y) = \sum_{\beta} \hat{A}_{\beta} \chi_{\beta}(y) \qquad \beta \subseteq [n]$$

$$A(xy) = \sum_{\gamma} \hat{A}_{\gamma} \chi_{\gamma}(xy). \qquad \gamma \subseteq [n].$$

Thus $\Pr[\text{Accept}]$

$$= \frac{1}{2} + \frac{1}{2}\mathop{\mathbb{E}}_{x,y}\left[\left(\sum_{\alpha} \hat{A}_{\alpha}\chi_{\alpha}(x)\right)\cdot\left(\sum_{\beta} \hat{A}_{\beta}\chi_{\beta}(y)\right)\cdot\left(\sum_{\gamma} \hat{A}_{\gamma}\chi_{\gamma}(xy)\right)\right]$$

$$= \frac{1}{2} + \frac{1}{2}\mathop{\mathbb{E}}_{x,y}\left[\sum_{\alpha,\beta,\gamma} \hat{A}_{\alpha}\hat{A}_{\beta}\hat{A}_{\gamma}\,\chi_{\alpha}(x)\,\chi_{\beta}(y)\,\chi_{\gamma}(xy)\right]$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{\alpha,\beta,\gamma} \hat{A}_\alpha \hat{A}_\beta \hat{A}_\gamma \mathop{\mathbb{E}}_{x,y} \left[\chi_\alpha(x) \chi_\beta(y) \underline{\chi_\gamma(xy)} \right]$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{\alpha,\beta,\gamma} \hat{A}_\alpha \hat{A}_\beta \hat{A}_\gamma \mathop{\mathbb{E}}_{x,y} \left[\chi_\alpha(x) \cdot \underline{\chi_\gamma(x)} \cdot \chi_\beta(y) \cdot \underline{\chi_\gamma(y)} \right]$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{\alpha,\beta,\gamma} \hat{A}_\alpha \hat{A}_\beta \hat{A}_\gamma \mathop{\mathbb{E}}_{x} \left[\chi_\alpha(x) \cdot \chi_\gamma(x) \right] \cdot \mathop{\mathbb{E}}_{y} \left[\chi_\beta(y) \cdot \chi_\gamma(y) \right].$$

The expectations are zero unless $\alpha = \gamma$, $\beta = \gamma$, and 1 otherwise.

<u>Hence</u>

$$\Pr[\text{Acc}] = \frac{1}{2} + \frac{1}{2} \sum_\alpha \hat{A}_\alpha^3$$

$$= \frac{1}{2} + \frac{1}{2} \sum_\alpha \hat{A}_\alpha^2 \cdot \hat{A}_\alpha \qquad \boxed{\sum_\alpha \hat{A}_\alpha^2 = 1}$$

$$\leq \frac{1}{2} + \frac{1}{2} \left(\max_{\alpha \subseteq [n]} \hat{A}_\alpha \right).$$

<u>Recall</u> $\quad \Delta(A, x_\alpha) \geq \delta \quad \forall\, \alpha \subseteq [n]$

$$\equiv \quad \frac{1}{2} - \frac{1}{2}\hat{A}_\alpha \geq \delta \quad \forall\, \alpha \subseteq [n]$$

$$\equiv \quad 1 - 2\delta \geq \hat{A}_\alpha \quad \forall\, \alpha \subseteq [n]$$

$$\equiv \quad \max_{\alpha \subseteq [n]} \hat{A}_\alpha \leq 1 - 2\delta .$$

Thus

$$\Pr[Acc] \leq \frac{1}{2} + \frac{1}{2}(1-2\delta) = 1 - \delta .$$

<u>To conclude</u>

① $\Delta(A, x_\alpha) \geq \delta \quad \forall\, \alpha \subseteq [n]$

$$\Longrightarrow \quad \Pr[Acc] \leq 1 - \delta .$$

② $\Pr[Acc] \geq \frac{1}{2} + \varepsilon$ then $\frac{1}{2} + \frac{1}{2}\sum_\alpha \hat{A}^3_\alpha \geq \frac{1}{2} + \varepsilon$

$\exists\, \alpha_0 \subseteq [n]$ s.t. $\hat{A}_{\alpha_0} \geq 2\varepsilon .$

Dictatorship Testing

Def: $Dict_i : \{-1,1\}^n \to \{-1,1\}$, $1 \le i \le n$, is the function

$$Dict_i(x_1, x_2, \cdots, x_n) = x_i.$$

$$\equiv \quad Dict_i = \chi_{\{i\}}, \quad \text{i.e. singleton character.}$$

Test Given $A : \{-1,1\}^n \to \{-1,1\}$, test whether it is a dictatorship.

- 3 queries, linear $A(x)\,A(y)\,A(z) = 1$.
- If $A = Dict_i$ then $\Pr[\text{Accept}] \geq 1 - \varepsilon$.
- If A is "far" from being a dictatorship, then $\Pr[\text{Accept}] \leq \frac{1}{2} + \varepsilon$.

(Contrapositive) If $\Pr[\text{Accept}] \geq \frac{1}{2} + \varepsilon$ then A "resembles" a dictatorship.

Fact A is dictatorship iff

$A = \chi_\alpha$ with $|\alpha| = 1$.

Def A is h-junta if (for now!)

$A = \chi_\alpha$ with $|\alpha| = h$.

Def A is (δ, h)-junta if

$\exists\, \alpha \subseteq [n]$ s.t. $|\alpha| \leq h$, $|\hat{A}_\alpha| \geq \delta$.

Fact Dictatorship is a $(1,1)$-junta.

$\therefore$ (δ, h)-junta is relaxed form of dictatorship ("resembles" dictatorship).

Theorem [Håstad]

There is a 3-query, linear test s.t.

① If A is a dictator, $\Pr[\text{Accept}] \geq 1-\varepsilon$.

② If $\Pr[\text{Accept}] \geq \frac{1}{2}+\eta$ then A is $(2\eta, O(1/\varepsilon \cdot \log(1/\eta)))$-junta.

Note If we just did the linearity test, $A(x)A(y) = A(xy)$ then

① holds, but

② fails, e.g. $\chi_{\{1,2,\ldots,n\}}$ passes with prob. 1 but is not a junta.

Need A mechanism to "punish" large size characters.

Add Noise!

<u>Def</u> Let $\mu \in \{-1,1\}^n$ be a noisy input with noise rate ε. I.e.

$\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, for $1 \le i \le n$, indep.

$$\mu_i = \begin{cases} +1 & \text{with prob. } 1-\varepsilon \\ -1 & \text{with prob. } \varepsilon. \end{cases}$$

<u>Observation</u>

- What is $\Pr_{\mu}[\underbrace{\chi_\alpha(\mu)}_{\prod_{i\in\alpha}\mu_i} = 1]$?

- Dict, $\alpha = \{j\}$, $\Pr_{\mu}[\mu_j = 1] = 1-\varepsilon$.

- In general,

$$\mathbb{E}_{\mu}[\chi_\alpha(\mu)] = \mathbb{E}_{\mu}\Big[\prod_{i\in\alpha}\mu_i\Big] = \prod_{i\in\alpha}\mathbb{E}[\mu_i] = (1-2\varepsilon)^{|\alpha|}$$

$$\therefore \Pr_{\mu}[\chi_\alpha(\mu)=1] = \mathbb{E}_{\mu}\Big[\frac{1+\chi_\alpha(\mu)}{2}\Big] = \frac{1}{2} + \frac{1}{2}(1-2\varepsilon)^{|\alpha|}$$

$$\to \frac{1}{2} \quad \text{as} \quad |\alpha| \to \infty.$$

Håstad Dictatorship Test

- Pick $x, y \in \{-1,1\}^n$ uniformly random.
- Pick $\mu \in \{-1,1\}^n$ as ε-noise.
- Accept iff

$$A(xy\underline{\mu}) = A(x) \cdot A(y)$$

$Z = xy\mu$

$Z_i = x_i y_i \mu_i$.

① Clearly, if A is dictatorship, say $\chi_{\{j\}}$

then

$$A(xy\mu) \stackrel{?}{=} A(x)\, A(y)$$

$$x_j y_j \mu_j \stackrel{?}{=} x_j y_j$$

$$\mu_j \stackrel{?}{=} +1$$

$\therefore$ Test accepts with prob. $1-\varepsilon$.

② Assume $\Pr[\text{Accept}] \geq \frac{1}{2} + \eta$. Then

$$\Pr[\text{Accept}] = \mathop{\mathbb{E}}_{x,y,\mu}\left[\frac{1 + A(x)\,A(y)\,A(xy\mu)}{2}\right]$$

$$\Pr[\text{Accept}] = \frac{1}{2} + \frac{1}{2} \sum_{\alpha} \hat{A}_{\alpha}^{3} \underbrace{(1-2\varepsilon)^{|\alpha|}}_{\mathop{\mathbb{E}}_{\mu}[\chi_{\alpha}(\mu)]}.$$

If $\Pr[\text{Accept}] \geqslant \frac{1}{2} + \eta$ then

$$\frac{1}{2} + \frac{1}{2} \sum_{\alpha} \hat{A}_{\alpha}^{3} (1-2\varepsilon)^{|\alpha|} \geqslant \frac{1}{2} + \eta$$

$$\therefore \quad \sum_{\alpha} A_{\alpha}^{3} (1-2\varepsilon)^{|\alpha|} \geqslant 2\eta$$

$$\therefore \quad \sum_{\alpha} A_{\alpha}^{2} \cdot \underline{A_{\alpha} (1-2\varepsilon)^{|\alpha|}} \geqslant 2\eta$$

Since $\sum_{\alpha} A_{\alpha}^{2} = 1$, we conclude

$\exists\, \alpha_0 \subseteq [n]$ s.t. $|\hat{A}_{\alpha_0}| (1-2\varepsilon)^{|\alpha_0|} \geqslant 2\eta$,

$\therefore$ $|\hat{A}_{\alpha_0}| \geqslant 2\eta$, $\quad |\alpha_0| \leq O(1/\varepsilon \log(1/\eta))$

$\therefore$ A is $(2\eta, O(1/\varepsilon \log(1/\eta)))$-junta. $\blacksquare$

Dictatorship as Encoding (Long Code).

- $\mathrm{Enc} : \{1, 2, \ldots, n\} \to \{-1, 1\}^{2^n}$.

- For $1 \le i \le n$,

  Long Code (i) = Table of values of Dict_i $(= \chi_{\{i\}})$.

- Dictatorship testing $\equiv$ Long Code testing.

—— x ——

Label Cover Hardness

Dictatorship (Long Code) testing

"compose"

$+$

Håstad's 3-bit PCP.