

# Håstad's 3-Bit PCP

Theorem NP has PCP verifier that

- uses  $O(\log n)$  random bits
- 3 queries, linear predicate (over  $\mathbb{F}_2$ )
- $x \in L \Rightarrow \exists \pi \Pr[\text{Accept}] \geq 1 - \epsilon$
- $x \notin L \Rightarrow \forall \pi \Pr[\text{Accept}] \leq \frac{1}{2} + \epsilon$

≡

Theorem Given 3-Lin instance  $S$

$$\begin{array}{c} \vdots \\ x \oplus y \oplus z = 0 \end{array}$$

$$x \oplus w \oplus u = 1$$

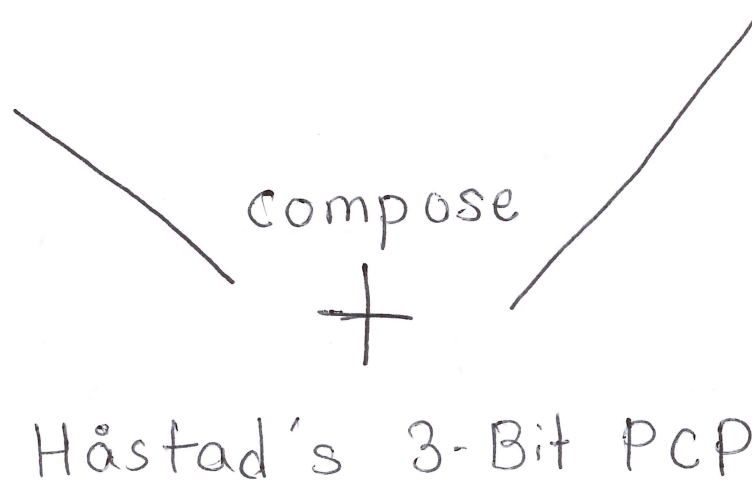
It is NP-hard to distinguish bet<sup>n</sup>

$$\text{(YES)} \quad \text{OPT}(S) \geq 1 - \epsilon$$

$$\text{(NO)} \quad \text{OPT}(S) \leq \frac{1}{2} + \epsilon.$$

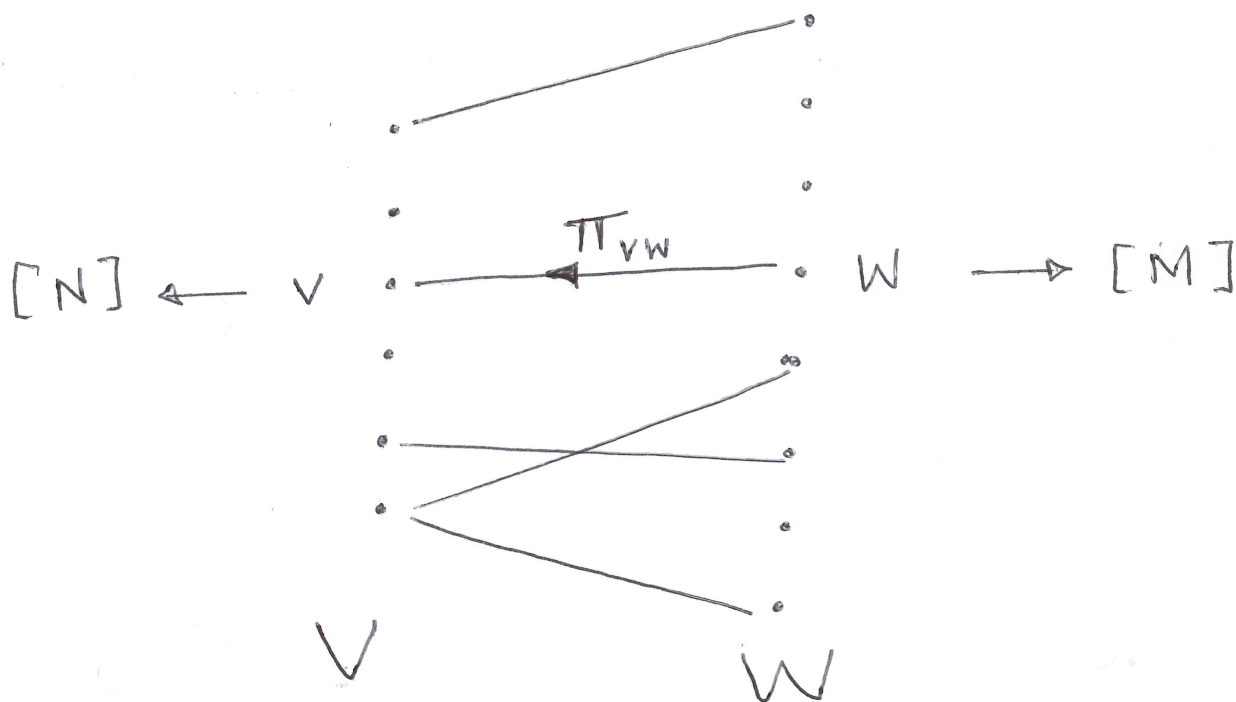
- PCP Theorem  
(Gap 3SAT)
- Label Cover
- Parallel Repetition  
Theorem

- $A: \{-1, 1\}^n \rightarrow \{-1, 1\}$
- Linearity Testing
- Dictatorship  
Testing



# Label Cover

$$\mathcal{L}(G(V, W, E), [N], [M], \{\pi_{vw}\}_{(v,w) \in E})$$



- $\pi_{vw} : [M] \rightarrow [N]$ . (projection).
- Labeling:  $l : W \rightarrow [M]$ ,  $l : V \rightarrow [N]$ .
- Labeling  $l$  "satisfies" edge  $(v, w)$  if

$$\pi_{vw}(l(w)) = l(v).$$

- Goal: Find a labeling that satisfies max fraction of edges.  $\text{OPT}(\mathcal{L})$ .

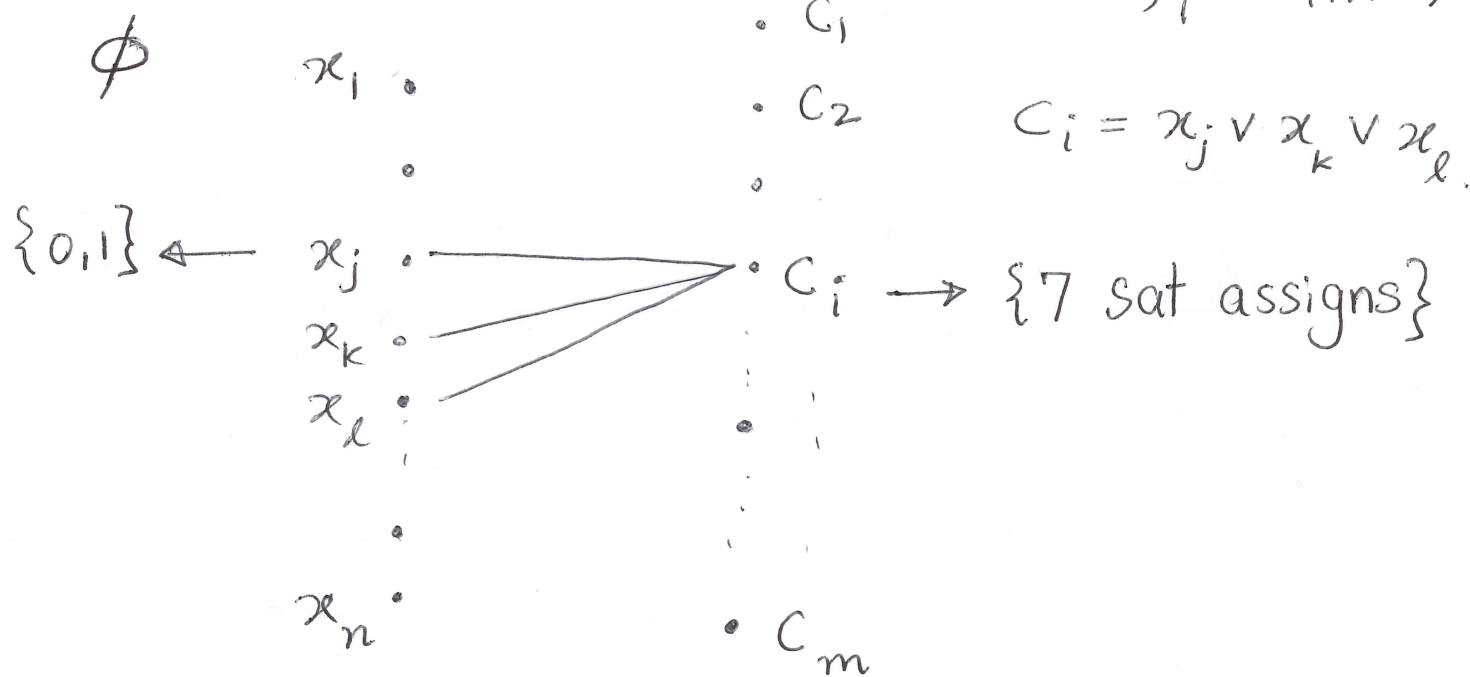
Theorem It is NP-hard to distinguish if

$\mathcal{L}(G(V, W, E), [2], [7], \{\pi_{vw}\})$  has

(YES)  $\text{OPT}(\mathcal{L}) = 1$  or

(NO)  $\text{OPT}(\mathcal{L}) \leq \alpha$  ( $\alpha < 1$ , constant).

Proof Reduction from  $\text{Gap3SAT}_{\alpha, \beta}$  (PCP Thm).



-  $\pi_{x_j, C_i}$  maps assign to  $C_i$  to that to  $x_j$ .

-  $\text{OPT}(\phi) = 1 \Rightarrow \text{OPT}(\mathcal{L}) = 1$ .

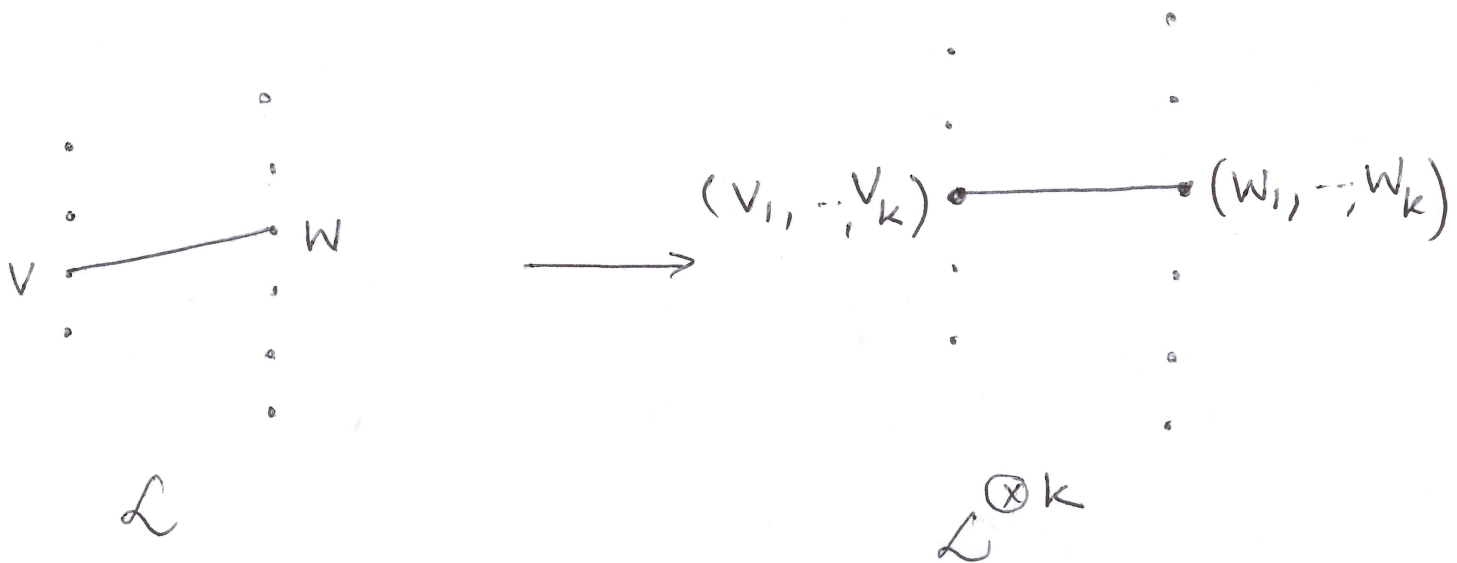
$\text{OPT}(\phi) \leq \beta \Rightarrow \text{OPT}(\mathcal{L}) \leq 1 - \frac{1-\beta}{3} = \alpha$ .

# Parallel Repetition

$$\mathcal{L}(G(V, W, E), [N], [M], \{\pi_{vw}\})$$

↓

$$\mathcal{L}^{\otimes k} = \mathcal{L}'(G'(V', W', E'), [N'], [M'], \{\pi_{v'w'}\})$$



$$- \text{OPT}(\mathcal{L}) = 1 \quad \Rightarrow \quad \text{OPT}(\mathcal{L}^{\otimes k}) = 1.$$

- [Raz]

$$\text{OPT}(\mathcal{L}) \leq \alpha \quad \Rightarrow \quad \text{OPT}(\mathcal{L}^{\otimes k}) \leq \tilde{\alpha}^k.$$

(if  $\alpha < 1$  then  $\tilde{\alpha} < 1$ ).

## Label Cover Hardness

It is NP-hard to distinguish if

$\mathcal{L}(G(V, W, E), [N], [M], \{\pi_{vw}\})$  has

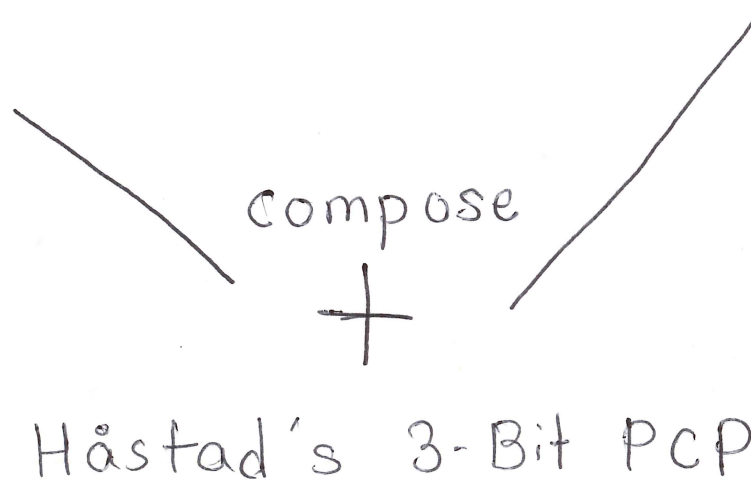
(YES)  $\text{OPT}(\mathcal{L}) = 1$  or

(NO)  $\text{OPT}(\mathcal{L}) \leq \epsilon = 2^{-k}$ .

Here,  $|G| = n^{O(k)}$ ,  $N, M$  are  $2^{O(k)}$ .

- PCP Theorem  
(Gap 3SAT)
- Label Cover
- Parallel Repetition  
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# Basics of Fourier Analysis

- $2^n$ -dim space of  $A: \{-1,1\}^n \rightarrow \mathbb{R}$
- Mainly interested in Boolean,  $A: \{-1,1\}^n \rightarrow \{-1,1\}$
- $A(x_1, x_2, \dots, x_n)$ ,  $x_i \in \{-1,1\}$ ,  $x = (x_1, \dots, x_n)$ .

Def For  $\alpha \subseteq [n]$ , character  $\chi_\alpha: \{-1,1\}^n \rightarrow \{-1,1\}$

$$\chi_\alpha(x) = \prod_{i \in \alpha} x_i.$$

$$\chi_\emptyset \equiv 1.$$

Def  $A, B: \{-1,1\}^n \rightarrow \mathbb{R}$ .

$$\langle A, B \rangle = \mathbb{E}_x [A(x) B(x)].$$

Def w.r.t.  $\langle, \rangle$ ,  $\{\chi_\alpha \mid \alpha \subseteq [n]\}$  forms an orthonormal basis.



Fact Every  $A: \{-1,1\}^n \rightarrow \mathbb{R}$  can be

written as

$$A(x) = \sum_{\alpha \subseteq [n]} \hat{A}_\alpha \chi_\alpha(x).$$

Fourier coefficients

Fact  $\hat{A}_\alpha = \langle A, \chi_\alpha \rangle.$

Fact  $\sum_{\alpha \subseteq [n]} \hat{A}_\alpha^2 = 1$  if  $A$  is Boolean.

Def  $A, B$  Boolean.  $\Delta(A, B) = \Pr_x [A(x) \neq B(x)].$

Fact  $\Delta(A, \chi_\alpha) = \frac{1}{2} - \frac{1}{2} \hat{A}_\alpha.$

Proof next page.

$$\hat{A}_d = \langle A, \chi_d \rangle$$

$$= \mathbb{E}_x [A(x) \chi_d(x)]$$

$$= \Pr_x [A(x) = \chi_d(x)] - \Pr_x [A(x) \neq \chi_d(x)]$$

$$= 1 - 2 \Pr_x [A(x) \neq \chi_d(x)]$$

$$= 1 - 2 \Delta(A, \chi_d).$$

$$\therefore \Delta(A, \chi_d) = \frac{1}{2} - \frac{1}{2} \hat{A}_d.$$

In particular,

$$\hat{A}_d \rightarrow 1 \iff \Delta(A, \chi_d) \rightarrow 0.$$