Homework I
G22.3033-002
Computational Complexity

February 23, 2009

You are expected to solve all the problems, but for grading purposes, submit written solutions to any 4 of the problems excluding problems 2 and 4. Due on Monday, March 9^{th}. Collaboration is allowed; please mention your collaborators.

1. **(2-SAT vs MAX-2SAT) :** An instance of 2-SAT consists of n boolean variables $x_1, x_2, \ldots, x_n$ and m clauses, and each clause contains at most 2 literals. We say that the instance is satisfiable if there is a {TRUE, FALSE}-assignment to $x_1, x_2, \cdots, x_n$ that satisfies every clause.

   - Show that deciding if an instance of 2-SAT is satisfiable is in P. *Hint: Given two clauses, one involving x_i and the other involving $\overline{x_i}$ try and replace them with a single clause.*
   - Show that deciding if there is an assignment that satisfies at least k out of m clauses is NP-complete (this problem is known as MAX-2SAT). Note that the parameter k is now part of the input. *Hint : Use a reduction from Vertex Cover.*

2. The class EXP is defined as

   $$\text{EXP} = \bigcup_k \text{DTIME}(2^{n^k})$$

   Show that NP $\subseteq$ EXP and co-NP $\subseteq$ EXP.

3. **(Decision vs Search) :** Show that if P = NP, there is a polynomial time algorithm to find a satisfying assignment to a 3-SAT formula if such an assignment exists.

4. Let BIPARTITE denote the language of all (undirected) graphs which are bipartite. Show that BIPARTITE $\in$ NL.

5. A directed graph is *strongly connected* if for every pair of vertices (u, v) there is a directed path from u to v in G. Show that the problem of deciding whether a graph is strongly connected is NL-complete.

6. The 0-1 knapsack problem is defined as follows: Let $\{a_i\}_{i=1}^n, b$ be positive integers (represented in binary). The knapsack problem asks whether there is an integer solution to

$$\sum_{i=1}^{n} a_i X_i = b \qquad\qquad X_i \in \{0, 1\}$$

We know that this problem is NP-complete.

Show that if we remove the constraints that $X_i \in \{0, 1\}$ and allow X_i's to be arbitrary (possibly negative) integers, then the problem is in P. In other words, deciding if the following equation has an integer solution is in P.

$$\sum_{i=1}^{n} a_i X_i = b$$

7. A problem A is NP-hard if there is a polynomial time reduction to it from some NP-complete problem (A itself need not be in NP).

   - Show that the following problem is NP-hard. Given a polynomial $P(X_1, \cdots, X_n)$ with integer coefficients, the problem is to decide whether the following equation has an integer solution :

   $$P(X_1, \cdots, X_n) = 0$$

   Hint : Show that in fact the problem is NP-hard for polynomials of degree 2, using a reduction from knapsack.

   - Can't we simply guess a solution if it exists and verify it ? Doesn't this mean that the problem is in NP ?

8. **(Padding) :** For a language $L \subseteq \{0, 1\}^*$, and a function $f(n)$ (assume that $f(n)$ is computable in time $O(f(n))$), let $L_f \subseteq \{0, 1, \#\}^*$ denote the following language :

$$L_f := \{x\#^{f(|x|)} \mid x \in L\}$$

- Suppose that $L \in \text{DTIME}(f(n))$. Then show that $L_f \in \text{DTIME}(O(n))$. Show similar results for non-deterministic time classes and deterministic space classes.
- Show that if $f(n)$ is a polynomial function, then $L \in \text{P}$ iff $L_f \in \text{P}$.
- Show that $\text{P} \neq \text{DSPACE}(O(n))$. *Hint : Assume an equality and arrive at a contradiction via suitable padding and the Deterministic Space Hierarchy Theorem.*
- Define the class NEXP as

$$\text{NEXP} := \cup_k \text{NTIME}(2^{n^k})$$

Prove that if $\text{P} = \text{NP}$ then $\text{EXP} = \text{NEXP}$.