# GA.3520: Honors Analysis of Algorithms

## Practice Problems

Some of these problems could be quite difficult and do not necessarily represent the difficulty level of the final exam.

### Problem

The Fibonacci sequence $\{F_k \mid k = 0, 1, 2, \ldots\}$ is defined as follows:

$$F_0 = 0, F_1 = 1, \quad \text{and} \quad \forall\, k \geq 2, \quad F_k = F_{k-1} + F_{k-2}.$$

1. Prove that $\forall\, k \geq 2, \quad 2F_k = F_{k+1} + F_{k-2}$.
2. Let $\phi > 0,\ \psi < 0$ be the roots of the quadratic equation $1 + x = x^2$. Prove that

   $$\forall\, k \geq 0, \quad F_k = \frac{1}{\sqrt{5}}(\phi^k - \psi^k) = \lfloor \frac{\phi^k}{\sqrt{5}} \rceil$$

   where $\lfloor z \rceil$ denotes the nearest integer to $z$.
3. We want to express a given positive integer $n$ as a sum of Fibonacci numbers using the minimum number of terms. Consider a recursive greedy algorithm that finds the largest index $k$ such that $F_k \leq n$, writes $n = F_k + n'$ and then recursively expresses $n'$ as a sum of Fibonacci numbers (the algorithm stops if $n' = 0$). Prove that this algorithm finds an expression for a given positive integer as a sum of Fibonacci numbers using the minimum number of terms.

### Problem

Given a set of intervals on a line, design a polynomial time greedy algorithm to select minimum number of intervals such that every interval overlaps with at least one of the selected intervals.

### Problem

You are given $k$ sorted lists with $n$ elements each. Design an algorithm, as efficient as possible, to merge all of them together into a single sorted list.

### Problem

Suppose there is a collection of $n$ intervals $\{[a_i, b_i] \mid i = 1, \ldots, n\}$ such that every two intervals in this collection overlap. Prove that all the intervals in the collection overlap, i.e. there is a point on the real line that belongs to all these intervals.

## Problem

Given a sequence of distinct integers $(a_1, a_2, \ldots, a_n)$, design an $O(n)$ time algorithm to find for each number $a_i$ the smallest $j$ such that $a_j > a_i$ and $i < j \leq n$, if such a $j$ exists. That is, for each $a_i$, find the first integer to its right which is larger than it.

## Problem

Consider a chess-board of size $2^k \times 2^k$ and remove any one its squares. Prove that the remaining board can be *tiled* with $L$-shaped tiles of size 3 (i.e. a tile with 3 squares arranged in $L$-shape). Use induction on $k$ and a strategy based on Divide and Conquer paradigm.

## Problem

Given a sequence of integers $(a_1, a_2, \ldots, a_n)$, each integer either positive or negative, give an $O(n \log n)$ time algorithm to find a shortest sub-sequence of consecutive integers $a_i, a_{i+1}, \ldots, a_j$ whose sum is at least a given integer $M$.