Homework II
CSCI-GA.3350-001
Computational Complexity

February 8, 2014

You are expected to solve all the problems. Due on Wed, Mar 12^{th}. Collaboration is allowed; please mention your collaborators.

1. Show that $\Sigma_k = \mathrm{NP}^{CIRCUIT-SAT_{k-1}}$.

   We define $L \in \Sigma_k$ if and only if there is a deterministic polynomial time verifier V such that

   $$x \in L \iff \exists y_1 \, \forall y_2 \cdots \; Q_k y_k \;\; V(x, y_1, \cdots, y_k) = 1$$

   where length of $y_1, y_2, \ldots, y_k$ is bounded by a polynomial in $|x|$.

2.
   - Show that $\mathrm{P}^{\mathrm{PSPACE}} = \mathrm{NP}^{\mathrm{PSPACE}} = \mathrm{PSPACE}$.
   - Show that if PH = PSPACE, then PH collapses to some finite level.
   - Can PH have a complete problem (complete under polynomial time reductions) ?

3. **(DP-completeness)** This problem studies the class DP (D stands for difference). A language $L \in DP$ if and only if there are languages $B \in \mathrm{NP}$ and $C \in \mathrm{coNP}$ so that $L = B \cap C$.

   - The problem SAT-UNSAT is defined as follows: Given pair of Boolean formulae (ϕ, ψ), decide if ϕ is satisfiable and ψ is unsatisfiable. Show that this problem is DP-complete (under polynomial time reductions).
   - A graph G is in HC-CRITICAL is G is not Hamiltonian but adding any edge to G will make it Hamiltonian. Show that HC-CRITICAL is in DP.

4.
   - Show that $\mathrm{NP}^{\mathrm{BPP}} \subseteq \mathrm{BPP}^{\mathrm{NP}}$ (*Hint : First show that a language in* $\mathrm{NP}^{\mathrm{BPP}}$ *is accepted by a polytime NTM that makes a single query to a* BPP *oracle and that too at the end*).
   - Show that if NP $\subseteq$ BPP, then PH collapses to BPP.

5. **(NEXP-completeness)** Define NEXP $= \cup_{k=1,2,\ldots}$ NTIME(2^{n^k}).

   Show that the following problem is NEXP-complete : Given $\langle M, x, n\rangle$, consisting of description of a NTM M, input x and an integer n in binary, decide if M has an accepting computation on x in n steps.

6. A circuit C is called an *implicit representation* of another circuit C^* if C takes as input a binary integer i such that $n+1 \leq i \leq N$, and outputs a triple $(TYPE, j, k)$ where

   - Input to C^* is an n-bit string $x_1 x_2 \ldots x_n$.
   - TYPE $\in \{\mathrm{AND}, \mathrm{OR}, \mathrm{NOT}\}$ indicates the type of i^{th} gate in circuit C^*.
   - $1 \leq j, k \leq N$.
   - The input of the i^{th} gate in C^* is the output of the j^{th} and k^{th} gates of C^* (if TYPE= NOT, then k is ignored. If $1 \leq j, k \leq n$, then the j^{th} or k^{th} gate is taken to be an input bit x_i).
   - The N^{th} gate in C^* is its output gate.

   Note that we could have $N = 2^n$, the circuit C could be of size poly$(\log N)$ = poly(n) and still implcitly represent a circuit C^* of size N (in short, a circuit can implicitly represent another circuit of size exponential in its own size).

   Let IMPLICIT CIRCUIT-SAT be the following problem : Given a circuit C that is an implicit representation of circuit C^*, decide if C^* is satisfiable. Show that this problem is NEXP-complete (*Hint : Use the regular structure of the circuit produced in Cook-Levin reduction*).

7.
   - Show that $\mathrm{NP}^{\mathrm{NP} \cap \mathrm{coNP}} = \mathrm{NP}$.
   - Generalize this to $\mathrm{NP}^{\Sigma_k \cap \Pi_k} = \Sigma_k$.

8. The problem Graph Consistency (GC) asks, for two given sets A and B of graphs, whether there exists a graph G such that every graph $g \in A$ is isomorphic to a (not necessarily induced) subgraph of G but each graph $h \in B$ is not isomorphic to any subgraph of G. Show that GC is in Σ_2.

9. Show that if $\Sigma_k = \Pi_k$ for some k, then $\mathrm{PH} = \Sigma_k$.