Technology Review's "Yearly Problem"
Burgess H Rhodes, XVIII, 1960

In the **Puzzle Corner** of the MIT alumni magazine *MIT Technology Review*, the "yearly problem" is offered annually:

> "How many integers from 1 to 100 can you form using the four digits of the year exactly once each; the operators +, −, × (multiplication), and / (division); and exponentiation? We desire solutions containing the minimum number of operators; among solutions having a given number of operators, those using the digits in order are desired. Parentheses may be used; they do not count as operators. A leading minus sign, however, does count as an operator."

How many (not necessarily different) numbers, some of which are integers, can be obtained by this process?

In the following discussion a **UNE** (**U**nsimplified **N**umerical **E**xpression) is an arithmetic expression defined as a character string of integers, created by using the four digits of a prescribed set **Y** exactly once each, the five binary operators $\{+, -, \times, /, \wedge\}$, and delimiters to define order of operations. For example with $\mathbf{Y} = \{1, 2, 3, 6\}$,

$$12+36, \quad 1-236, \quad 1\times([2\times3]-6), \quad 12/(3+6), \quad [1\times2]\wedge[3/6], \quad (1-2)\wedge[3/6], \quad \text{and} \quad 1/([2\times3]-6)$$

(and many more) are all **UNE**s. Evaluations yield positive and negative integers, zero, rational, irrational, complex, and undefined values. Each of these might be preceded by a "−" to change sign. The "∧" is used to denote exponentiation:

$$1 \wedge 236 \equiv 1^{236}.$$

Digits in a **UNE** need not occur in any particular order. One pair of delimiters is introduced with each binary operation, except the last. Two **UNE**s are different if their character strings, including delimiter positions, are different.

About how many **UNE**s are there? Ignore an "attached" leading "−". Denote by N_i, $i = 1, 2, 3, 4$, an integer with i digits (in which, except for N_1, the leading digit cannot be a 0), and by O a binary operator. Then **before** the order of operations is specified, the possible **UNE**s take one of the following **forms**:

$$N_4, \quad N_3ON_1, \quad N_2ON_2, \quad N_1ON_3, \quad N_2ON_1ON_1, \quad N_1ON_2ON_1, \quad N_1ON_1ON_2, \quad \text{or} \quad N_1ON_1ON_1ON_1.$$

Suppose that **Y** consists of four **different**, **non-zero** digits. Then by form, the **UNE**s occur in the following numbers of ways (some obvious symmetries could be used to shorten the table):

	Count for Number of			
Form	Integers (each = 4!)	Operators	Op Orders	Total
N_4	4!	-	-	24
N_3ON_1	$\binom{4}{3}\cdot 3!\cdot\binom{1}{1}$	5	1	120
N_2ON_2	$\binom{4}{2}\cdot 2!\cdot\binom{2}{2}\cdot 2!$	5	1	120
N_1ON_3	$\binom{4}{1}\cdot\binom{3}{3}\cdot 3!$	5	1	120
$N_2ON_1ON_1$	$\binom{4}{2}\cdot 2!\cdot\binom{2}{1}\cdot\binom{1}{1}$	5^2	2!	1200
$N_1ON_2ON_1$	$\binom{4}{1}\cdot\binom{3}{2}\cdot 2!\cdot\binom{1}{1}$	5^2	2!	1200
$N_1ON_1ON_2$	$\binom{4}{1}\cdot\binom{3}{1}\cdot\binom{2}{2}\cdot 2!$	5^2	2!	1200
$N_1ON_1ON_1ON_1$	$\binom{4}{1}\cdot\binom{3}{1}\cdot\binom{2}{1}\cdot\binom{1}{1}$	5^3	$3!-1$	15000

for a grand total 18,984, an **upper bound** on the number of different **UNE**s. Note that the **UNE**s corresponding to orders of operation $\{1, 3, 2\}$ and $\{2, 3, 1\}$ imposed on $N_1ON_1ON_1ON_1$ are identical, with common value $(N_1ON_1)O(N_1ON_1)$. The upper bound is smaller if **Y** contains repeated digits and/or 0's.

The upper bound on number of possible **UNE**s increases dramatically, to 2,038,128, if an optional leading "$-$" can be attached to each initial N_i and to the completion of each binary operation. Such attachments would increase the number of operators in **UNE**s, leading to less desirable solutions to the "yearly problem."

As an experiment, and with computer assistance*, some of the **UNE**s for $\mathbf{Y} = \{1, 2, 3, 6\}$ were generated. Delimiters for $+$ and $-$ are parentheses "( )" and delimiters for $\times$ and $/$ are square brackets "[ ]". Exponents are placed in exponent position. A total of 87 different integers in [1,100] were generated. For each integer a "best" **UNE** is given, selected first for fewest operations, and then for largest number of digits in natural position (there are often many **UNE**s with the same "score"):

87 Different Integers in [1,100] Generated from Digits in Year 1236

$1 = 1^{236}$	$30 = 1 + 23 + 6$	$61 = 1 \times (63 - 2)$
$2 = 32/16$	$31 = 62 - 31$	$62 = 1 + 63 - 2$
$3 = 36/12$	$32 = (1 + 63)/2$	$63 = 1^2 \times 63$
$4 = (1 + 23)/6$	$33 = 1^6 + 32$	$64 = 16^{[3/2]}$
$5 = 31 - 26$	$34 = [6/2] + 31$	$65 = (62 + 3) \times 1$
$6 = 12 \times [3/6]$	$35 = 1 - 2 + 36$	$66 = 1 + 2 + 63$
$7 = 23 - 16$	$36 = 1^2 \times 36$	$67 = 6^2 + 31$
$8 = [12/6]^3$	$37 = 1^2 + 36$	$68 = 136/2$
$9 = 12 + 3 - 6$	$38 = 61 - 23$	$69 = [12 \times 6] - 3$
$10 = [12/3] + 6$	$39 = 13 + 26$	$70 = 3^2 + 61$
$11 = 16 - 3 - 2$	$40 = (6^2 + 3) + 1$	$71 = [2 \times 36] - 1$
$12 = 21 - 3 - 6$	$42 = 126/3$	$72 = 216/3$
$13 = 26 - 13$	$43 = 31 + [2 \times 6]$	$73 = 1 + [2 \times 36]$
$14 = 12 + [6/3]$	$45 = 3 \times (21 - 6)$	$74 = 2 \times (1 + 36)$
$15 = 36 - 21$	$46 = [16 \times 3] - 2$	$75 = 12 + 63$
$16 = 32 - 16$	$47 = [6 \times 2^3] - 1$	$76 = [13 \times 6] - 2$
$17 = 1 \times (23 - 6)$	$48 = 12 + 36$	$77 = 13 + 2^6$
$18 = 6/(1 - [2/3])$	$49 = 62 - 13$	$78 = 1 \times [26 \times 3]$
$19 = 1 + [36/2]$	$50 = [16 \times 3] + 2$	$79 = 1 + [26 \times 3]$
$20 = [13 \times 2] - 6$	$51 = 63 - 12$	$80 = 2 + [13 \times 6]$
$21 = 12 + 3 + 6$	$52 = 312/6$	$81 = (1 + 26) \times 3$
$22 = 132/6$	$53 = 61 - 2^3$	$82 = 1 + (3 + 6)^2$
$23 = 1 \times (26 - 3)$	$54 = 162/3$	$84 = 21 + 63$
$24 = 36 - 12$	$55 = 61 - [2 \times 3]$	$87 = 261/3$
$25 = 32 - 1 - 6$	$56 = 61 - 3 - 2$	$90 = (12 + 3) \times 6$
$26 = (16 - 3) \times 2$	$57 = 21 + 36$	$93 = 62 + 31$
$27 = (3^6)^{[1/2]}$	$58 = 62 - 3 - 1$	$95 = 2^6 + 31$
$28 = 6 + 23 - 1$	$59 = (62 - 3) \times 1$	$96 = 3 \times [2 \times 16]$
$29 = 61 - 32$	$60 = 62 - 3 + 1$	$100 = ((1 + 6) + 3)^2$

* My program in HTBasic, version 9.0, from the TransEra Corporation, Draper, Utah, running on my Dell Workstation 340, vintage 2002.

A total 6025 integers in [1,100] are generated using digits of the year 1236:

Frequency Distribution (N) by Value (V) for UNEs for Year 1236

V	N	V	N	V	N	V	N	V	N	V	N	V	N	V	N
1	871	14	72	27	87	40	12	53	3	66	44	79	4	92	0
2	241	15	142	28	25	41	0	54	62	67	38	80	8	93	12
3	218	16	99	29	51	42	32	55	5	68	15	81	53	94	0
4	392	17	49	30	94	43	2	56	7	69	8	82	4	95	2
5	249	18	134	31	8	44	0	57	6	70	4	83	0	96	17
6	409	19	20	32	21	45	4	58	3	71	2	84	4	97	0
7	212	20	64	33	29	46	3	59	11	72	32	85	0	98	0
8	170	21	74	34	24	47	2	60	26	73	4	86	0	99	0
9	326	22	15	35	32	48	53	61	33	74	8	87	1	100	6
10	133	23	27	36	205	49	11	62	31	75	14	88	0		
11	157	24	120	37	32	50	6	63	16	76	2	89	0		
12	215	25	16	38	67	51	2	64	75	77	4	90	8		
13	40	26	23	39	82	52	8	65	49	78	24	91	0		

The product of the non-zero N's in this table is an estimate of number of possible solutions (without regard to "scoring") to the 1236 "yearly problem."

Thus the value 1 can be represented in 871 ways, too many to list conveniently for illustration. But value 30 can be represented in 94 ways, as shown in the following table:

94 UNEs with Value 30 from Digits of Year 1236

$30 = (1+23)+6$
$30 = (1+26)+3$
$30 = (6+23)+1$
$30 = 3+(21+6)$
$30 = (26+3)+1$
$30 = 23+(1+6)$
$30 = [3\times 12]-6$
$30 = (3+6)+21$
$30 = 6+(3+21)$
$30 = [1\times(2+3)]\times 6$
$30 = (3+[2/1])\times 6$
$30 = 6\times(2+3^1)$
$30 = 6\times[(2+3)/1]$
$30 = (2+3)\times[1\times 6]$
$30 = (2+[3/1])\times 6$
$30 = ([3/1]+2)\times 6$
$30 = 6\times(2^1+3)$
$30 = [(2+3)/1]\times 6$
$30 = (2+3)\times[6/1]$
$30 = 3\times[(6-1)\times 2]$
$30 = 6\times(3^1+2)$
$30 = 6/[1/(2+3)]$
$30 = [(2+3)\times 6]^1$
$30 = [3\times(6-1)]\times 2$
$30 = 1+(23+6)$
$30 = (1+3)+26$
$30 = 1+(26+3)$
$30 = 6+(23+1)$
$30 = (3+26)+1$
$30 = 3+(1+26)$
$30 = (21+6)+3$
$30 = 26+(1+3)$
$30 = [3\times 6]+12$
$30 = (2+[1\times 3])\times 6$
$30 = ([1\times 3]+2)\times 6$
$30 = 6\times(2+[3/1])$
$30 = [6\times(2+3)]\times 1$
$30 = (2+3)/[1/6]$
$30 = (3+[1\times 2])\times 6$
$30 = 1\times[6\times(2+3)]$
$30 = 6\times([1\times 3]+2)$
$30 = [(6-1)\times 3]\times 2$
$30 = (6-1)\times[2\times 3]$
$30 = 6\times(3+2^1)$
$30 = 6\times([1\times 2]+3)$
$30 = 6^1\times(2+3)$
$30 = [(6-1)\times 2]\times 3$
$30 = [6/1]\times(2+3)$
$30 = 12+[3\times 6]$
$30 = (21+3)+6$
$30 = 1+(3+26)$
$30 = (1+6)+23$
$30 = (6+21)+3$
$30 = 3+(26+1)$
$30 = (23+6)+1$
$30 = 3+(6+21)$
$30 = ([1\times 2]+3)\times 6$
$30 = (2^1+3)\times 6$
$30 = ([2/1]+3)\times 6$
$30 = 6\times([2\times 3]-1)$
$30 = [6\times(2+3)]/1$
$30 = (2+3)^1\times 6$
$30 = (3^1+2)\times 6$
$30 = 3\times[2\times(6-1)]$
$30 = 6\times([2/1]+3)$
$30 = [1\times 6]\times(2+3)$
$30 = 2\times[(6-1)\times 3]$
$30 = 6\times(3+[1\times 2])$
$30 = 6\times([3/1]+2)$
$30 = [(2+3)\times 6]\times 1$
$30 = [2\times(6-1)]\times 3$
$30 = [12\times 3]-6$
$30 = (3+21)+6$
$30 = 21+(3+6)$
$30 = (23+1)+6$
$30 = 1+(6+23)$
$30 = 6+(21+3)$
$30 = (26+1)+3$
$30 = 6+(1+23)$
$30 = 1\times[(2+3)\times 6]$
$30 = (3+2^1)\times 6$
$30 = 6\times(2+3)^1$
$30 = 6\times[(2+3)\times 1]$
$30 = [6\times(2+3)]^1$
$30 = (2+3^1)\times 6$
$30 = ([2\times 3]-1)\times 6$
$30 = 6\times(2+[1\times 3])$
$30 = [(2+3)\times 1]\times 6$
$30 = (2+3)\times 6^1$
$30 = 2\times[3\times(6-1)]$
$30 = 6\times(3+[2/1])$
$30 = 6\times[1\times(2+3)]$
$30 = [(2+3)\times 6]/1$
$30 = [2\times 3]\times(6-1)$