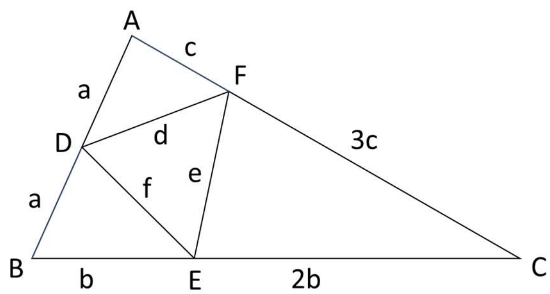
Problem M/J2, MIT News May/June, 2021

Based on the problem statement, notations for the length of the sides of the triangle are added to the figure.



Using the cosine formula for determining the magnitude of the angles in a triangle given the length of its sides, the following relationships are obtained.

$$\frac{a^2+c^2-d^2}{2ac}=\frac{4a^2+16c^2-9b^2}{16ac}$$

$$\frac{a^2+b^2-f^2}{2ab}=\frac{4a^2+9b^2-16c^2}{12ab}$$

$$\frac{4b^2+9c^2-e^2}{12bc}=\frac{9b^2+16c^2-4a^2}{24bc}$$

Solve for a^2, b^2 and c^2 in terms of d^2, e^2 and f^2.

$$\begin{bmatrix}4 & 9 & -8\\ 2 & -3 & 16\\ 4 & -1 & 2\end{bmatrix}\begin{Bmatrix}a^2\\ b^2\\ c^2\end{Bmatrix}=\begin{bmatrix}8 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{Bmatrix}d^2\\ e^2\\ f^2\end{Bmatrix}$$

$$\begin{Bmatrix}a^2\\ b^2\\ c^2\end{Bmatrix}=\begin{bmatrix}\frac{4}{25} & -\frac{3}{25} & \frac{12}{25}\\ \frac{24}{25} & \frac{12}{25} & -\frac{8}{25}\\ \frac{4}{25} & \frac{12}{25} & -\frac{3}{25}\end{bmatrix}\begin{Bmatrix}d^2\\ e^2\\ f^2\end{Bmatrix}$$

In terms of the length of its sides, the area of triangle DEF is given by:

$$A_{DEF}=\frac{\sqrt{2d^2f^2+2d^2e^2+2e^2f^2-d^4-e^4-f^4}}{4}$$

Similarly for triangle ABC

$$A_{ABC}=$$

$$\frac{\sqrt{72a^2b^2+128a^2c^2+288b^2c^2-16a^4-81b^4-256c^4}}{4}$$

Replacing a^2, b^2 and c^2 with their expressions in terms of d^2, e^2 and f^2 yields:

$$A_{ABC}=$$

$$\frac{\sqrt{\begin{array}{c}72\left(\frac{4}{25}d^2-\frac{3}{25}e^2+\frac{12}{25}f^2\right)\left(\frac{24}{25}d^2+\frac{12}{25}e^2-\frac{8}{25}f^2\right)\\ +128\left(\frac{4}{25}d^2-\frac{3}{25}e^2+\frac{12}{25}f^2\right)\left(\frac{4}{25}d^2+\frac{12}{25}e^2-\frac{3}{25}f^2\right)\\ +288\left(\frac{24}{25}d^2+\frac{12}{25}e^2-\frac{8}{25}f^2\right)\left(\frac{4}{25}d^2+\frac{12}{25}e^2-\frac{3}{25}f^2\right)\\ -16\left(\frac{4}{25}d^2-\frac{3}{25}e^2+\frac{12}{25}f^2\right)^2\\ -81\left(\frac{24}{25}d^2+\frac{12}{25}e^2-\frac{8}{25}f^2\right)^2\\ -256\left(\frac{4}{25}d^2+\frac{12}{25}e^2-\frac{3}{25}f^2\right)^2\end{array}}}{4}$$

Factoring, multiplying out the terms, and grouping like terms yields:

$$A_{ABC}=\frac{30\sqrt{2d^2e^2+2d^2f^2+2e^2f^2-d^4-e^4-f^4}}{25}$$

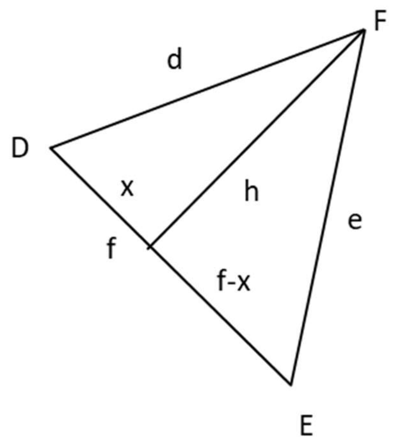
$$A_{ABC}=\frac{6(4)\sqrt{2d^2e^2+2d^2f^2+2e^2f^2-d^4-e^4-f^4}}{5(4)}$$

$$A_{ABC}=\frac{24}{5}\frac{\sqrt{2d^2e^2+2d^2f^2+2e^2f^2-d^4-e^4-f^4}}{4}$$

The right term is the expression for A_{DEF}. Thus:

$$A_{ABC}=\frac{24}{5}A_{DEF}$$

If needed, the area of a triangle based on the length of its sides is developed as follows.



Determine the height (h) in terms for the side lengths.

$$h^2 + x^2 = d^2$$

$$h^2 + (f - x)^2 = e^2$$

Subtract the first expression from the second expression and solve for x. Use the result and the first expression to solve for h.

$$f^2 - 2fx = e^2 - d^2$$

$$x = \frac{d^2 + f^2 - e^2}{2f}$$

$$h^2 = d^2 - \left(\frac{d^2 + f^2 - e^2}{2f}\right)^2$$

$$h^2 = \frac{4d^2f^2}{4f^2} - \frac{d^4 + f^4 + e^4 + 2d^2f^2 - 2d^2e^2 - 2e^2f^2}{4f^2}$$

$$h^2 = \frac{2d^2f^2 + 2d^2e^2 + 2e^2f^2 - d^4 - e^4 - f^4}{4f^2}$$

$$h = \frac{\sqrt{2d^2f^2 + 2d^2e^2 + 2e^2f^2 - d^4 - e^4 - f^4}}{2f}$$

The area of the triangle is:

$$A_{DEF} = \frac{1}{2}fh = \frac{\sqrt{2d^2f^2 + 2d^2e^2 + 2e^2f^2 - d^4 - e^4 - f^4}}{4}$$