Problem M/A 3 from Puzzle Corner – Technology Review, 2020

Burgess H Rhodes, XVIII, 1960

The Problem.

"Our last regular problem is a venture into astronomy/3D geometry from Lee Giesecke. Imagine it's the winter solstice and you want to compute the geocentric and geodetic latitudes of the Arctic Circle and the Tropic of Capricorn. The geocentric latitude gives the angle between the equatorial plane and a line from Earth's center to a point on the surface. The geodetic latitude assumes a line from the same surface point that is perpendicular to a plane tangent to Earth's surface. The angle of intersection of this line with the equatorial plane gives the geodetic latitude. Assume the Earth can be represented by an ellipsoid of revolution with the semi-major and semi-minor axes $a = 1$ and $b = 0.99665$. Assume that Earth's obliquity (axial tilt with respect to the plane of the ecliptic) is $23.44°$."

The Model. Define additional symbols

$$\begin{aligned} \tau &= \text{ axial tilt,} \\ \theta &= \text{ geocentric latitude, and} \\ \theta' &= \text{ geodetic latitude.} \end{aligned}$$

A planar slice through the north and south poles of the earth model is an ellipse similar to

$$\left(\frac{x}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = 1, \qquad (\mathbf{E})$$

with x in the equatorial plane, z on the polar axis, and $(0, 0)$ the ellipse center. Let $P_o = (x_o, z_o)$ be a point on this ellipse.

Geocentric Latitude. The geocentric latitude of P_o is

$$\theta_o = \tan^{-1}\left(\frac{z_o}{x_o}\right).$$

Geodetic Latitude. Differentiate (**E**) implicitly to obtain

$$\frac{2}{a^2}x + \frac{2}{b^2}z\frac{dz}{dx} = 0$$

from which obtain

$$\frac{dz}{dx} = \frac{-\,b^2x}{a^2z}.$$

The negative reciprocal

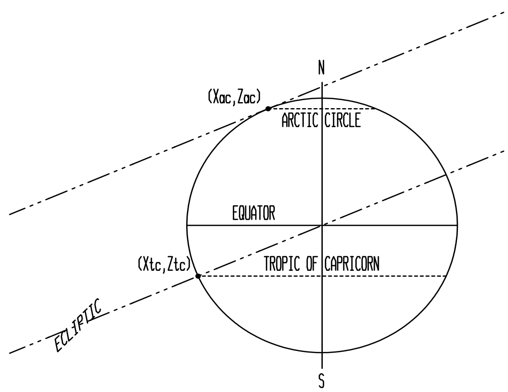
$$-\frac{dx}{dz} = \frac{a^2z}{b^2x}$$

is the slope of the line through (x, z) perpendicular to the tangent line to the ellipse (and thus perpendicular to the tangent plane to the earth model). The geodetic latitude of P_o is

$$\theta'_o = \tan^{-1}\left(\frac{a^2z_o}{b^2x_o}\right).$$

Note: both latitudes are angles between $0°$ and $90°$ measured from the equatorial plane, with an appended N or S to distinguish the Northern and Southern hemispheres.

The Earth Model at Winter Solstice
$(a = 1, b = 0.99665, \tau = 23.44°)$



Latitudes for the Arctic Circle. Now assume it is the time of the winter solstice. A point P_{ac} on the Arctic Circle is a point on ellipse (**E**) at which the tangent line has slope τ, the axial tilt. Thus solve the system

$$\left(\frac{x}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = 1, \text{ and}$$
$$\left[\frac{dz}{dx} =\right] \frac{-b^2x}{a^2z} = \tan\tau$$

for $P_{ac} = (x_{ac}, z_{ac})$:

$$x_{ac} = \frac{-a^2\tan\tau}{\sqrt{b^2 + a^2\tan^2\tau}}, \quad z_{ac} = \frac{b^2}{\sqrt{b^2 + a^2\tan^2\tau}}.$$

Adjusted for northern hemisphere, the latitudes for the Arctic Circle are

$$\theta_{ac} = \tan^{-1}\left(\frac{z_{ac}}{x_{ac}}\right) = \tan^{-1}\left(\frac{b^2}{a^2\tan\tau}\right) = \tan^{-1}\left(\frac{0.99665^2}{\tan(23.44°)}\right) \equiv 66.42°N$$
$$\theta'_{ac} = \tan^{-1}\left(\frac{a^2z_{ac}}{b^2x_{ac}}\right) = \tan^{-1}\left(\frac{1}{\tan\tau}\right) = \tan^{-1}\left(\frac{1}{\tan(23.44°)}\right) \equiv 66.56°N$$

Latitudes for the Tropic of Capricorn. A point P_{tc} on the Tropic of Capricorn is a point on ellipse (**E**) through which a line with slope τ passes through the ellipse center. Thus solve the system

$$\left(\frac{x}{a}\right)^2 + \left(\frac{z}{b}\right)^2 = 1, \text{ and}$$
$$z = \tan\tau \cdot x$$

for $P_{tc} = (x_{tc}, z_{tc})$:

$$x_{tc} = \frac{a \cdot b}{\sqrt{b^2 + a^2\tan^2\tau}}, \quad z_{tc} = \frac{a \cdot b \cdot \tan\tau}{\sqrt{b^2 + a^2\tan^2\tau}}.$$

Adjusted for southern hemisphere, the latitudes for the Tropic of Capricorn are

$$\theta_{tc} = \tan^{-1}\left(\frac{z_{tc}}{x_{tc}}\right) = \tan^{-1}(\tan\tau) = \tan^{-1}(\tan(23.44°)) \equiv 23.44°S$$
$$\theta'_{tc} = \tan^{-1}\left(\frac{a^2z_{tc}}{b^2x_{tc}}\right) = \tan^{-1}\left(\frac{a^2\tan\tau}{b^2}\right) = \tan^{-1}\left(\frac{\tan(23.44°)}{0.99665^2}\right) \equiv 23.58°S$$