Problem M/A 2 from Puzzle Corner – Technology Review, 2019

Burgess H Rhodes, XVIII, 1960

The Problem.

"Nob Yashigahara wants you to place the digits 1 through 9 once each into the nine boxes to yield a valid equation."

$$\frac{\square}{\square\,\square} + \frac{\square}{\square\,\square} + \frac{\square}{\square\,\square} = 1$$

Solutions. Let U, V, and W be three different digits in the range 1 to 9, and let $Q = \dfrac{\square}{\square\,\square} \equiv \dfrac{U}{VW}$.

Interpretation **A**: If the denominators $\square\,\square$ are meant to denote a 10's digit followed by a 1's digit, then relevant fractions are $Q = \dfrac{U}{VW} = \dfrac{U}{10 \times V + W}$ and a solution is:

$$\frac{7}{68} + \frac{5}{34} + \frac{9}{12} = 1.$$

Interpretation **B**: If the denominators are meant to denote the product of two digits, then relevant fractions are $Q = \dfrac{U}{V \times W}$ and a solution is:

$$\frac{1}{3 \times 6} + \frac{5}{8 \times 9} + \frac{7}{2 \times 4} = 1.$$

Variation **C**: If the denominators are changed to denote the sum of two digits, then relevant fractions are $Q = \dfrac{U}{V + W}$ and solutions are:

$$\frac{2}{6+9} + \frac{3}{7+8} + \frac{4}{1+5} = 1,$$
$$\frac{2}{7+8} + \frac{3}{6+9} + \frac{4}{1+5} = 1, \text{ and}$$
$$\frac{1}{2+4} + \frac{5}{7+8} + \frac{6}{3+9} = 1.$$

Naturally additional solutions are obtained in all cases by permuting the three fractions. And in cases **B** and **C** the digits in the denominators in each fraction can be interchanged, leading technically to additional solutions.

Method of Solution. A simple search with some efficiencies was used in all cases:

(1) Establish an array $D(*)$ of all fractions Q for which $Q < 1$ (automatic for interpretation **A**). For cases **B** and **C** apply the additional requirement that $V < W$. Each row of the array $D(*)$ holds four numbers: U, V, W, and Q. For **A** there are $9 \cdot 8 \cdot 7 = 504$ rows, for **B** 218 rows, and for **C** 202 rows.

(2) Reorder the rows of $D(*)$ so that the Q's are in increasing order.

(3) Let N be the number of rows in $D(*)$, and use i, j, and k as row indices. Then

  (a) For $i = 1$ to $N - 2$,

  (b) For $j = i + 1$ to $N - 1$, exit j-level search if $Q_i + Q_j > 1$, otherwise

  (c) For $k = j + 1$ to N do the following:

$$\text{if} \quad Q_i + Q_j + Q_k \begin{cases} < 1 & \text{continue search,} \\ = 1 & \text{test for possible solution, and} \\ > 1 & \text{exit } k\text{-level search.} \end{cases}$$

The test for a possible solution is the simple check that $U_i, V_i, W_i, U_j, V_j, W_j, U_k, V_k, W_k$ are all different.