What follows is an analysis for problem M/A 2; the only solution is $\frac{9}{12}+\frac{5}{34}+\frac{7}{68}=1$. The analysis involves elementary number theory and finding upper bounds for sums over classes of possible solutions. After a series of steps we show that the numerators must be $5,7,9$ while the denominators must include $1*, 3*$. This is followed by more case studies to eliminate all cases but the solution cited.

Some notation from elementary number theory: $a|b$ means a divides b (a, b are integers), for example $5|30$. An observation about fractions that is used without comment below is that when positive integers r, s, t, u satisfy $r > t$ and $s < u$ we have $\frac{r}{s}+\frac{t}{u} > \frac{t}{s}+\frac{r}{u}$, i.e. a larger sum results from pairing the larger numerator with the smaller denominator.

Problem M/A 2 may be restated as finding a solution for

$$S=\frac{a}{A}+\frac{b}{B}+\frac{e}{E} \quad = \quad 1 \text{ for } a,b,e \in [1,9], A,B,E \in [12,98], \qquad (1)$$

with the added constraints that a, b, e, A, B, E share no common digits and do not end in 0. Imposing the above constraints, Eq. 1 is equivalent to

$$aBE + bAE + eAB \quad = \quad ABE. \qquad (2)$$

The minimal denominator must be ≤ 19, as $\frac{9}{21}+\frac{9}{31}+\frac{9}{41} < 1$. Thus $1*$ must appear in some denominator and 1 appears in no numerator. Observe that the minimal denominator cannot be $17, 18, 19$ as the sum in Eq. 1 would then be bounded above by $\frac{9}{17}+\frac{8}{23}+\frac{5}{46} < 1$ and $\frac{9}{18}+\frac{7}{23}+\frac{6}{45} < 1$ and $\frac{8}{19}+\frac{7}{23}+\frac{6}{45} < 1$ respectively.

The first claim is that $\{A, B, E\}$ contains no denominator ending in 5, i.e. divisible by 5. For example if $5|A$, Eq. 2 shows that $5|aBE$, hence $5|a$ or $5|B$ or $5|E$, contrary to the assumption of no common digits.

Now observe that no denominator contains 9. First 9 cannot be the lead digit, as otherwise the maximum sum would have denominators $12, 34, 96$ or $13, 24, 96$ or $14, 23, 96$ or $16, 23, 94$ and the corresponding expressions would be: $\frac{8}{12}+\frac{7}{34}+\frac{5}{96}$ or $\frac{8}{13}+\frac{7}{24}+\frac{5}{96}$ or $\frac{8}{14}+\frac{7}{23}+\frac{5}{96}$ or $\frac{8}{16}+\frac{7}{23}+\frac{5}{94}$, respectively; but all these sums are too small.

We now show no denominator ends in 9. Assume A ends in 9 and is prime. Eq. 2 shows $p|B, E$ and we may assume $p|B$. If $A = 29$ we have $B = 58, 87$ and the maximal expressions in Eq. 1 would be $\frac{6}{29}+\frac{4}{58}+\frac{7}{13}$ or $\frac{5}{29}+\frac{4}{87}+\frac{6}{13}$, but both sums are too small. The other prime numerators ending in 9 are $59, 79, 89$ and are too big to allow $A|B$. The remaining cases are for composite $A = 39, 49, 69$. For $A = 39$ the maximal expressions in Eq. 1 would contain the fraction $\frac{8}{1*}$ and would be bounded above by $\frac{7}{39} + \frac{6}{45} + \frac{8}{12}$ or $\frac{5}{39} + \frac{7}{26} + \frac{8}{14}$ or $\frac{5}{39} + \frac{7}{24} + \frac{8}{16}$ that are all too small. A similar analysis for $A = 49$ reveals upper bounds are $\frac{6}{49} + \frac{7}{35} + \frac{8}{12}$ or $\frac{5}{49} + \frac{7}{26} + \frac{8}{13}$ and for $A = 69$ upper bounds are $\frac{5}{69} + \frac{7}{34} + \frac{8}{12}$ or $\frac{5}{69} + \frac{7}{24} + \frac{8}{13}$, but all sums are too small.

Thus 9 must appear in a numerator.

We now show $A = 13$ does not yield a solution. Eq. 2 shows $13|B$ or $13|E$; we may assume the former so $B = 26, 52, 78$. As $\frac{9}{13} + \frac{5}{26} + \frac{4}{78} < 1$ and $\frac{9}{13} + \frac{6}{52} + \frac{4}{78} < 1$ we cannot have $13|E$. Substitution of $B = 26, 52, 78$ in (2) and division by 13 shows $26me = (26m - 2am - b)E$ for $m = 1, 2, 3$ respectively. For $m = 1$ we see $E = 2e$ and $13 = 26 - 2a - b$ that fails since $E \geq 34$. For $m = 2$ we see $52 - 4a - b = 13r = 13, 26, 39$ and $4e = rE$ for $r = 1, 2, 3$. As the unused digits are $4, 6, 7, 8, 9$ we see $E \geq 46$ so $4e = rE$ fails. For $m = 3$ we see $78 - 6a - b = 13r = 26, 39, 52, 65$ and $6e = Er$ for $r = 2, 3, 4, 5$. As the unused digits are $2, 4, 5, 6, 9$ we see $E = 24$ or $E \geq 42$. The only candidate is $(E, e, r) = (24, 8, 2)$. This fails as $B = 78$ has an 8.

Thus we may assume $A = 12, 14, 16$.

We now claim no 7 appears in a denominator. We first show 7 cannot appear as the lead digit. The cases $E = 73, 74, 76, 78$ are easily disposed of as they contain a prime factor $p \geq 13$, i.e. $73, 37, 19, 13$ respectively. Eq. 2 shows $p|A$ or $p|B$. For $E = 73$ we need $B = 73$; for $E = 74$ we need $B = 37$; both fail due to repeated 7. For $E = 76$ we need $B = 38, 57$; 57 fails due to repeated 7. $B = 38$ implies $A(2b + e) = 4 * 19(A - a)$ by substitution in (2) and division by 38. This tells us that $2b + e = 19$ and $A = 4(A - a)$. The latter relation implies $(a, A) = (9, 12)$. However $2b + e = 19$ fails with unused digits $4, 5$. For $E = 78$ we need $B = 26, 52$, but both fail since $A \geq 14$ ($A = 12$ is disallowed as B uses 2) and $\frac{9}{14} + \frac{5}{26} + \frac{3}{78} < 1$.

The final case with lead digit 7 in a denominator is $E = 72$, so that $A = 14, 16$. $A = 14$ fails since $\frac{9}{14} + \frac{8}{36} + \frac{5}{72} < 1$. Similarly $A = 16$ fails since $\frac{9}{16} + \frac{8}{34} + \frac{5}{72} < 1$.

We now show 7 cannot be the trailing digit in a denominator: the candidates are $27, 37, 47, 57, 67, 87$ (17 was eliminated earlier). The cases $E = 37, 47, 57, 67, 87$ are easily disposed of as they contain a prime factor $p \geq 13$, i.e. $37, 47, 19, 67, 29$ respectively. We must have $p|B$ (note $A = 19$ was eliminated above). There are no remaining viable choices for B when $p = 37, 47, 67, 29$. For $E = 57$ we must have $B = 38$ and this fails as $\frac{9}{12} + \frac{6}{38} + \frac{4}{57} < 1$. The final case is $E = 27$, with $A = 14, 16$. Eq. 2 shows $27|eB$, hence $3|B$. If $A = 14$ Eq. 2 shows $7|B$ (since $a = 7$ is disallowed) with remaining digits $3,5,6,8,9$; thus $B = 63$ (only multiple of 21) and $e = 9$ is forced by $27|eB$. This fails as $\frac{8}{14} + \frac{5}{63} + \frac{9}{27} < 1$. If $A = 16$ Eq. 2 shows $16|aB$, so $2|B$; earlier we saw $3|B$, so $6|B$. With remaining digits $3,4,5,8,9$ we must have $B = 48, 54, 84$. $B = 48$ and $B = 84$ force $e = 9$ since $27|eB$, but $\frac{5}{16} + \frac{3}{48} + \frac{9}{27} < 1$ so both fail. $B = 54$ forces $a = 8$ since $16|aB$, but $\frac{8}{16} + \frac{3}{54} + \frac{9}{27} < 1$ so this fails.

Thus 7 must appear in a numerator.

We now show 3 is not a trailing digit in a denominator. It suffices to consider the cases $E = 23, 43, 53, 63, 83$. The cases $E = 23, 43, 53, 83$ are easily disposed of as they are all prime ≥ 23, so we have $E|B$. This is impossible for $E = 53, 83$. For $E = 23$ we must have $B = 46$ and now no digits remain for A. For $E = 43$ we must have $B = 86$ that forces $A = 12$, but even $\frac{9}{12} + \frac{7}{43} + \frac{5}{86} < 1$.

The final case to show 3 is not a trailing digit in a denominator is $E = 63$ with $A = 12, 14$. Eq. 2 shows $63|eAB$, so $21|eB, 9|eB$ respectively. When $A = 12$ we have either $e = 7$ or $B = 84$. If $(A, e) = 12, 7$ we have $B = 48, 54, 58, 84$ but $T = \frac{9}{12} + \frac{5}{48} + \frac{7}{63} < 1$ so we must have $B = 54$ as 58 is not a multiple of 3 and T is an upper bound when $B = 84$. Substitution in Eq. 2 and division by $6 * 7 * 9 = 378$ gives $9a + 2b = 96$ with $\{a, b\} = \{8, 9\}$ and this fails. If $e \neq 7$ and $(A, B, E) = (12, 84, 63)$ we see $\frac{9}{12} + \frac{7}{84} + \frac{5}{63} < 1$ so $(A, B, E) = (12, 84, 63)$ fails. When $A = 14$ the remaining digits are $2, 5, 7, 8, 9$, so $B = 28, 52, 58, 82$. Since $9|eB$ we must have $e = 9$ (3 divides none of the choices for B) but $\frac{7}{14} + \frac{5}{28} + \frac{9}{63} < 1$ so this fails.

Thus 3 is not a trailing digit in a denominator.

We observed earlier that 5 cannot be the trailing digit in a denominator. We now show 5 is not a leading digit either. It suffices to consider the cases $E = 52, 54, 56, 58$. The cases $E = 52, 58$ are easily disposed of as they contain a prime factor $p \geq 13$, i.e. 13, 29 respectively that forces $p|B$ but there are no such candidates for B using the digits left.

When $E = 54$ we must have $A = 12, 16$. As $\frac{9}{16} + \frac{8}{27} + \frac{3}{54} < 1$ we must have $A = 12$ and $B = 36, 38, 68, 86$. Substitution of $(A, E) = (12, 54)$ in Eq. 2 and division by 6 shows $9aB + 108b + 2eB = 108B$, so $B|108b$. The latter condition forces $B = 36$ as $B = 38, 68, 86$ have prime factors ≥ 17. We now have $\{a, b, e\} = \{7, 8, 9\}$ and $9a + 3b + 2e = 108$, so $3|e$, i.e. $e = 9$ and $9a + 3b = 90$ that cannot be solved as we need $3|b$.

The final case to consider is $E = 56$, so $A = 12, 14$. Substitution of $(A, E) = (12, 56)$ in Eq. 2 and division by 4 shows $14aB + 168b + 3eB = 168B$. We also have $B = 34, 38, 48, 84$ using the remaining digits $3, 4, 7, 8, 9$. As

$B|168b$ we must have $B = 48, 84$ as $B = 34, 38$ have prime factors ≥ 17. $B = 48, 84$ both fail as $\frac{9}{12} + \frac{3}{56} + \frac{7}{48} < 1$. We are left with $(A, E) = (14, 56)$ and substitution in Eq. 2 and division by 14 shows $4aB + 56b + eB = 56B$. With remaining digits $2, 3, 7, 8, 9$ we have $B = 28, 32, 38, 82$. As $B|56b$ we must have $B = 28, 32$ as $B = 38, 82$ have prime factors ≥ 19. $B = 28$ fails as $\frac{9}{14} + \frac{7}{28} + \frac{3}{56} < 1$. If $B = 32$ we see $b = 8$ (since $B|56b$) and also $7|(4a + e)B$, i.e. $7|(4a + e)$ with $\{a, e\} = \{7, 9\}$ that fails.

Thus 5 appears only as a numerator, and $\{a, b, e\} = \{5, 7, 9\}$ while the denominators are even $\{A = 1*, B = 3*, E\}$ with $A = 1* \in \{12, 14, 16\}$.

If $A = 16$ and $a \leq 7$ the sum in Eq. 1 would then be bounded above by $\frac{7}{16} + \frac{9}{23} + \frac{8}{54} < 1$. For $a = 9$ we must have $E \in \{24, 28\}$; otherwise the sum is bounded above by $\frac{9}{16} + \frac{7}{32} + \frac{5}{48} < 1$. We also have $B \in \{32, 34, 38\}$. From Eq. 2 we derive $16(bE + eB) = 7BE$, so $16|BE$. The only solution is $(E, B) = (24, 38)$; upon substitution and division by 16 we have $24b + 38e = 7 * 3 * 19$ that is even on the left and odd on the right.

Now consider $A = 14$. As $\frac{7}{14} + \frac{9}{32} + \frac{5}{68} < 1$ we must have $E = 26, 28$ when $a \leq 7$. As $\frac{5}{14} + \frac{9}{26} + \frac{7}{38} < 1$ there are no solutions for $a = 5$. For $a = 7$ Eq. 1 gives us $2(bE + eB) = BE$. If $E = 26$ we see $13|B = 3*$ has no solutions. If $E = 28$ we have $28b + eB = 14B$ that implies $7|B = 3*$ but is unsolvable.

To address the final case for $A = 14$, we have $a = 9$ and Eq. 1 gives us $14(bE + eB) = 5BE$ with unused digits $2, 3, 5, 6, 7, 8$. As $7|BE$ we must have $(E, B) = (28, 36)$ based on the remaining digits. This implies $28b + 36e = 360$, so $36|28b$, i.e. $9|7b$ and $b = 9$ that is disallowed as $a = 9$.

The final case is $A = 12$. As $\frac{7}{12} + \frac{9}{34} + \frac{5}{68} < 1$ there are no solutions for $a \leq 7$. For $(a, A) = (9, 12)$ we have $4(bE + eB) = EB$ from Eq. 1 with unused digits $3, 4, 5, 6, 7, 8$. Note $E > B = 3*$ and $B \in \{34, 36, 38\}$.

If $B = 38$ we see $19|E$ with $E = 46, 64$ that fails. If $B = 36$ we have $E = 48, 84$, but $\frac{9}{12} + \frac{5}{36} + \frac{7}{48} > 1$ so $B \neq 48$. As $7|84 = E$ we have $e = 7$ and so $b = 5$ but $4(5*84+7*36) \not\equiv 36*84 \bmod 5$ since the left side is $4*2 \bmod 5$ and the right is $4 \bmod 5$.

If $B = 34$ we have $E = 68, 86$. As $17|B$ we must have $E = 68$. As $\frac{9}{12} + \frac{5}{34} + \frac{7}{68} = 1$, this the only solution.