What follows is an analysis for problem M/A 2; the only solution is \( \frac{9}{12} + \frac{5}{34} + \frac{7}{68} = 1 \). The analysis involves elementary number theory and finding upper bounds for sums over classes of possible solutions. After a series of steps we show that the numerators must be 5, 7, 9 while the denominators must include 1*, 3*. This is followed by more case studies to eliminate all cases but the solution cited.

Some notation from elementary number theory: \( a \mid b \) means \( a \) divides \( b \) \((a, b \) are integers), for example 5|30. An observation about fractions that is used without comment below is that when positive integers \( r, s, t, u \) satisfy \( r > t \) and \( s < u \) we have \( \frac{r}{s} + \frac{t}{u} > \frac{t}{s} + \frac{r}{u} \), i.e. a larger sum results from pairing the larger numerator with the smaller denominator.

Problem M/A 2 may be restated as finding a solution for

\[
S = \frac{a}{A} + \frac{b}{B} + \frac{e}{E} = 1 \quad \text{for } a, b, e \in [1, 9], \ A, B, E \in [12, 98],
\]

with the added constraints that \( a, b, e, A, B, E \) share no common digits and do not end in 0. Imposing the above constraints, Eq. 1 is equivalent to

\[
aBE + bAE + eAB = ABE.
\]

The minimal denominator must be \( \leq 19 \), as \( \frac{9}{21} + \frac{9}{31} + \frac{9}{41} < 1 \). Thus 1* must appear in some denominator and 1 appears in no numerator. Observe that the minimal denominator cannot be 17, 18, 19 as the sum in Eq. 1 would then be bounded above by \( \frac{9}{17} + \frac{8}{23} + \frac{5}{46} < 1 \) and \( \frac{9}{18} + \frac{7}{23} + \frac{6}{45} < 1 \) and \( \frac{8}{19} + \frac{7}{23} + \frac{6}{43} < 1 \) respectively.

The first claim is that \( \{A, B, E\} \) contains no denominator ending in 5, i.e. divisible by 5. For example if 5|A, Eq. 2 shows that 5|aBE, hence 5|a or 5|B or 5|E, contrary to the assumption of no common digits.

Now observe that no denominator contains 9. First 9 cannot be the lead digit, as otherwise the maximum sum would have denominators 12, 34, 96 or 13, 24, 96 or 14, 23, 96 or 16, 23, 94 and the corresponding expressions would be: \( \frac{8}{12} + \frac{7}{34} + \frac{5}{96} \) or \( \frac{8}{13} + \frac{7}{24} + \frac{5}{96} \) or \( \frac{8}{14} + \frac{7}{23} + \frac{5}{96} \) or \( \frac{8}{16} + \frac{7}{23} + \frac{5}{94} \), respectively; but all these sums are too small.

We now show no denominator ends in 9. Assume A ends in 9 and is prime. Eq. 2 shows \( p|B, E \) and we may assume \( p|B \). If \( A = 29 \) we have \( B = 58, 87 \) and the maximal expressions in Eq. 1 would be \( \frac{5}{29} + \frac{4}{58} + \frac{7}{13} \) or \( \frac{5}{29} + \frac{4}{87} + \frac{6}{13} \), but both sums are too small. The other prime numerators ending in 9 are 59, 79, 89 and are too big to allow \( A|B \). The remaining cases are
for composite $A = 39, 49, 69$. For $A = 39$ the maximal expressions in Eq. 1 would contain the fraction $\frac{8}{15}$ and would be bounded above by $\frac{7}{39} + \frac{6}{45} + \frac{8}{12}$ or $\frac{5}{39} + \frac{7}{26} + \frac{8}{14}$ or $\frac{5}{39} + \frac{7}{21} + \frac{8}{16}$ that are all too small. A similar analysis for $A = 49$ reveals upper bounds are $\frac{6}{49} + \frac{7}{35} + \frac{8}{12}$ or $\frac{5}{49} + \frac{7}{26} + \frac{8}{13}$ and for $A = 69$ upper bounds are $\frac{5}{69} + \frac{7}{34} + \frac{8}{12}$ or $\frac{5}{69} + \frac{7}{24} + \frac{8}{13}$, but all sums are too small.

Thus 9 must appear in a numerator.

We now show $A = 13$ does not yield a solution. Eq. 2 shows $13|B$ or $13|E$; we may assume the former so $B = 26, 52, 78$. As $\frac{9}{13} + \frac{5}{26} + \frac{4}{78} < 1$ and $\frac{9}{13} + \frac{6}{26} + \frac{4}{78} < 1$ we cannot have $13|E$. Substitution of $B = 26, 52, 78$ in (2) and division by 13 shows $26me = (26m - 2am - b)E$ for $m = 1, 2, 3$ respectively. For $m = 1$ we see $E = 2e$ and $13 = 26 - 2a - b$ that fails since $E \geq 34$. For $m = 2$ we see $52 - 4a - b = 13r = 13, 26, 39$ and $4e = rE$ for $r = 1, 2, 3$. As the unused digits are 4, 6, 7, 8, 9 we see $E \geq 46$ so $4e = rE$ fails. For $m = 3$ we see $78 - 6a - b = 13r = 26, 39, 52, 65$ and $6e = Er$ for $r = 2, 3, 4, 5$. As the unused digits are 2, 4, 5, 6, 9 we see $E = 24$ or $E \geq 42$. The only candidate is $(E, e, r) = (24, 8, 2)$. This fails as $B = 78$ has an 8.

Thus we may assume $A = 12, 14, 16$.

We now claim no 7 appears in a denominator. We first show 7 cannot appear as the lead digit. The cases $E = 73, 74, 76, 78$ are easily disposed of as they contain a prime factor $p \geq 13$, i.e. $73, 37, 19, 13$ respectively. Eq. 2 shows $p|A$ or $p|B$. For $E = 73$ we need $B = 73$; for $E = 74$ we need $B = 37$; both fail due to repeated 7. For $E = 76$ we need $B = 38, 57$; 57 fails due to repeated 7. $B = 38$ implies $A(2b + e) = 4 \times 19(A - a)$ by substitution in (2) and division by 38. This tells us that $2b + e = 19$ and $A = 4(A - a)$. The latter relation implies $(a, A) = (9, 12)$. However $2b + e = 19$ fails with unused digits 4, 5. For $E = 78$ we need $B = 26, 52$, but both fail since $A \geq 14$ ($A = 12$ is disallowed as $B$ uses 2) and $\frac{9}{14} + \frac{5}{26} + \frac{3}{78} < 1$.

The final case with lead digit 7 in a denominator is $E = 72$, so that $A = 14, 16$. $A = 14$ fails since $\frac{8}{14} + \frac{8}{36} + \frac{5}{72} < 1$. Similarly $A = 16$ fails since $\frac{9}{16} + \frac{8}{34} + \frac{5}{72} < 1$.

We now show 7 cannot be the trailing digit in a denominator: the candidates are $27, 37, 47, 57, 67, 87$ (17 was eliminated earlier). The cases $E = 37, 47, 57, 67, 87$ are easily disposed of as they contain a prime factor $p \geq 13$, i.e. $37, 47, 19, 67, 29$ respectively. We must have $p|B$ (note $A = 19$ was eliminated above). There are no remaining viable choices for $B$ when $p = 37, 47, 67, 29$. For $E = 57$ we must have $B = 38$ and this fails as $\frac{9}{12} + \frac{6}{38} + \frac{4}{57} < 1$. The final case is $E = 27$, with $A = 14, 16$. Eq. 2 shows $27|eB$, hence $3|B$. If $A = 14$ Eq. 2 shows $7|B$ (since $a = 7$ is disallowed) with
remaining digits 3, 5, 6, 8, 9; thus $B = 63$ (only multiple of 21) and $e = 9$ is forced by $27|eB$. This fails as $\frac{8}{14} + \frac{5}{63} + \frac{9}{27} < 1$. If $A = 16$ Eq. 2 shows $16|aB$, so $2|B$; earlier we saw $3|B$, so $6|B$. With remaining digits 3, 4, 5, 8, 9 we must have $B = 48, 54, 84$. $B = 48$ and $B = 84$ force $e = 9$ since $27|eB$, but $\frac{5}{16} + \frac{3}{48} + \frac{9}{27} < 1$ so both fail. $B = 54$ forces $a = 8$ since $16|aB$, but $\frac{8}{16} + \frac{3}{54} + \frac{9}{27} < 1$ so this fails.

Thus 7 must appear in a numerator.

We now show 3 is not a trailing digit in a denominator. It suffices to consider the cases $E = 23, 43, 53, 63, 83$. The cases $E = 23, 43, 53, 83$ are easily disposed of as they are all prime $\geq 23$, so we have $E|B$. This is impossible for $E = 53, 83$. For $E = 23$ we must have $B = 46$ and now no digits remain for $A$. For $E = 43$ we must have $B = 86$ that forces $A = 12$, but even $\frac{9}{12} + \frac{7}{43} + \frac{5}{86} < 1$.

The final case to show 3 is not a trailing digit in a denominator is $E = 63$ with $A = 12, 14$. Eq. 2 shows $63|eAB$, so $21|eB, 9|eB$ respectively. When $A = 12$ we have either $e = 7$ or $B = 84$. If $(A, e) = 12, 7$ we have $B = 48, 54, 58, 84$ but $T = \frac{9}{12} + \frac{5}{48} + \frac{7}{63} < 1$ so we must have $B = 54$ as 58 is not a multiple of 3 and $T$ is an upper bound when $B = 84$. Substitution in Eq. 2 and division by $6 \times 7 \times 9 = 378$ gives $9a + 2b = 96$ with $\{a, b\} = \{8, 9\}$ and this fails. If $e \neq 7$ and $(A, B, E) = (12, 84, 63)$ we see $\frac{9}{12} + \frac{7}{84} + \frac{5}{63} < 1$ so $(A, B, E) = (12, 84, 63)$ fails. When $A = 14$ the remaining digits are 2, 5, 7, 8, 9, so $B = 28, 52, 58, 82$. Since $9|eB$ we must have $e = 9$ (3 divides none of the choices for $B$) but $\frac{7}{14} + \frac{5}{28} + \frac{9}{63} < 1$ so this fails.

Thus 3 is not a trailing digit in a denominator.

We observed earlier that 5 cannot be the trailing digit in a denominator. We now show 5 is not a leading digit either. It suffices to consider the cases $E = 52, 54, 56, 58$. The cases $E = 52, 58$ are easily disposed of as they contain a prime factor $p \geq 13$, i.e. 13, 29 respectively that forces $p|B$ but there are no such candidates for $B$ using the digits left.

When $E = 54$ we must have $A = 12, 16$. As $\frac{9}{16} + \frac{8}{27} + \frac{3}{54} < 1$ we must have $A = 12$ and $B = 36, 38, 68, 86$. Substitution of $(A, E) = (12, 54)$ in Eq. 2 and division by 6 shows $9aB + 108b + 2eB = 108B$, so $B|108b$. The latter condition forces $B = 36$ as $B = 38, 68, 86$ have prime factors $\geq 17$. We now have $\{a, b, e\} = \{7, 8, 9\}$ and $9a + 3b + 2e = 108$, so $3|e$, i.e. $e = 9$ and $9a + 3b = 90$ that cannot be solved as we need $3|b$.

The final case to consider is $E = 56$, so $A = 12, 14$. Substitution of $(A, E) = (12, 56)$ in Eq. 2 and division by 4 shows $14aB + 168b + 3eB = 168B$. We also have $B = 34, 38, 48, 84$ using the remaining digits 3, 4, 7, 8, 9. As
B|168b we must have $B = 48, 84$ as $B = 34, 38$ have prime factors $\geq 17$. $B = 48, 84$ both fail as $\frac{9}{12} + \frac{3}{56} + \frac{7}{38} < 1$. We are left with $(A, E) = (14, 56)$ and substitution in Eq. 2 and division by 14 shows $4aB + 56b + eB = 56B$. With remaining digits 2, 3, 7, 8, 9 we have $B = 28, 32, 38, 82$. As $B|56b$ we must have $B = 28, 32$ as $B = 38, 82$ have prime factors $\geq 19$. $B = 28$ fails as $\frac{9}{14} + \frac{7}{28} + \frac{3}{48} < 1$. If $B = 32$ we see $b = 8$ (since $B|56b$) and also $7|(4a + e)B$, i.e. $7|(4a + e)$ with $\{a, e\} = \{7, 9\}$ that fails.

Thus 5 appears only as a numerator, and $\{a, b, e\} = \{5, 7, 9\}$ while the denominators are even {\{7, 16\}} giving us $2(\sum$ is bounded above by $B$ unused digits $2$, $a$ for $\{7, 16\}$, i.e. $7 \not| 168$). From Eq. 2 we derive $16(bE + eB) = 7BE$, so $16|BE$. The only solution is $(E, B) = (24, 38)$; upon substitution and division by 16 we have $24b + 38e = 7 \times 3 \times 19$ that is even on the left and odd on the right.

Now consider $A = 14$. As $\frac{7}{14} + \frac{9}{32} + \frac{5}{68} < 1$ we must have $E = 26, 28$ when $a \leq 7$. As $\frac{5}{14} + \frac{9}{26} + \frac{7}{38} < 1$ there are no solutions for $a = 5$. For $a = 7$ Eq. 1 gives us $2(bE + eB) = BE$. If $E = 26$ we see $13|B = 3*$ has no solutions. If $E = 28$ we have $28b + eB = 14B$ that implies $7|B = 3*$ but is unsolvable.

To address the final case for $A = 14$, we have $a = 9$ and Eq. 1 gives us $14(bE + eB) = 5BE$ with unused digits $2, 3, 5, 6, 7, 8$. As $7|BE$ we must have $(E, B) = (28, 36)$ based on the remaining digits. This implies $28b + 36e = 360$, so $36|28b$, i.e. $9|7b$ and $b = 9$ that is disallowed as $a = 9$.

The final case is $A = 12$. As $\frac{7}{12} + \frac{9}{34} + \frac{5}{68} < 1$ there are no solutions for $a \leq 7$. For $(a, A) = (9, 12)$ we have $4(bE + eB) = EB$ from Eq. 1 with unused digits $3, 4, 5, 6, 7, 8$. Note $E > B = 3*$ and $B \in \{34, 36, 38\}$.

If $B = 38$ we see $19|E$ with $E = 46, 64$ that fails. If $B = 36$ we have $E = 48, 84$, but $\frac{9}{12} + \frac{5}{36} + \frac{7}{48} > 1$ so $B \not= 48$. As $7|84 = E$ we have $e = 7$ and so $b = 5$ but $4(5 \times 84 + 7 \ast 36) \not= 36 \ast 84 \mod 5$ since the left side is $4 \ast 2 \mod 5$ and the right is $4 \mod 5$.

If $B = 34$ we have $E = 68, 86$. As $17|B$ we must have $E = 68$. As $\frac{9}{12} + \frac{5}{34} + \frac{7}{68} = 1$, this the only solution.