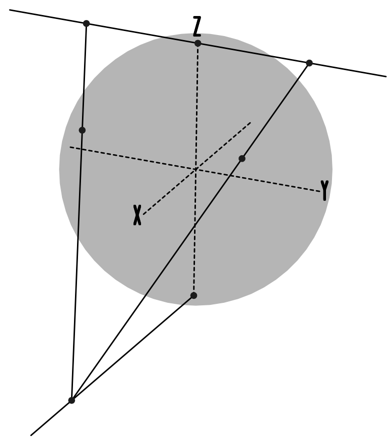
**Problem J/A3 from Puzzle Corner – Technology Review, 2019**

Burgess H Rhodes, XVIII, 1960

**The Problem:**

> "Robert Bird offers a problem for which a good 3-D geometric sense is very useful. A sphere of radius 1 has its center at the origin of an $x$-$y$-$z$ system. Line $L_1$ lies in the $y$-$z$ plane, is parallel to the $y$-axis, and is tangent to the sphere at $P_1 = (0, 0, 1)$. Line $L_2$ lies in the $x$-$z$ plane, is parallel to the $x$-axis, and is tangent to the sphere at $P_2 = (0, 0, -1)$. Line $L_3$ leaves $L_2$ at $P_3 = (x, 0, -1)$, is tangent to the sphere at $P_4$, and intersects $L_1$ at $P_5 = (0, y, 1)$. You are to find $y$ in terms of $x$."

**The Geometry**




**Solution Plan.** Line $L_3$ can be represented parametrically, with parameter $\lambda$, $-\infty < \lambda < \infty$, as

$$L_3(\lambda) = P_3 + \lambda(P_5 - P_3) = (x, 0, -1) + \lambda((0, y, 1) - (x, 0, -1)) = ((1 - \lambda)x, \lambda y, 2\lambda - 1).$$

The distance $D(\lambda)$ between a point on $L_3$ and the origin is defined by

$$D^2(\lambda) = [(1 - \lambda)x]^2 + [\lambda y]^2 + [2\lambda - 1]^2 = (x^2 + y^2 + 4)\lambda^2 - 2(x^2 + 2)\lambda + (x^2 + 1).$$

Point $P_4$, at distance 1 from the origin $(0, 0, 0)$, is the point on line $L_3$ which is closest to the origin. Steps to the problem solution are:

- **(a)** Minimize $D(\lambda)$: determine $\lambda^*$ for which $D(\lambda^*) \leq D(\lambda)$ for all $\lambda$ (routine calculus).
- **(b)** Solve $D(\lambda^*) = 1$ for $y \equiv y(x)$ (routine algebra).

**Step (a).** All distances $D(\lambda)$ of interest are positive. Thus to find $\lambda^*$ it suffices to minimize $D^2(\lambda)$:

$$\frac{d}{d\lambda} D^2(\lambda) = 2\lambda(x^2 + y^2 + 4) - 2(x^2 + 2)$$

which is 0 for

$$\lambda^* = \frac{x^2 + 2}{x^2 + y^2 + 4} = \frac{x^2 + 2}{(x^2 + 2) + (y^2 + 2)}.$$

**Step (b).** Equation $D(\lambda^*) = 1$ is equivalent to $D^2(\lambda^*) = 1$. Thus solve

$$(x^2 + y^2 + 4)(\lambda^*)^2 - 2(x^2 + 2)\lambda^* + (x^2 + 1) = 1.$$

Substitute

$$U = x^2 + 2 \quad V = y^2 + 2$$

to obtain

$$(U+V)\left(\frac{U}{U+V}\right)^2 - 2U\frac{U}{U+V} + (U-1) = 1.$$

Solve for $V$:

$$V = \frac{2U}{U-2},$$

producing

$$y^2 + 2 = \frac{2x^2+4}{x^2},$$

and final answer

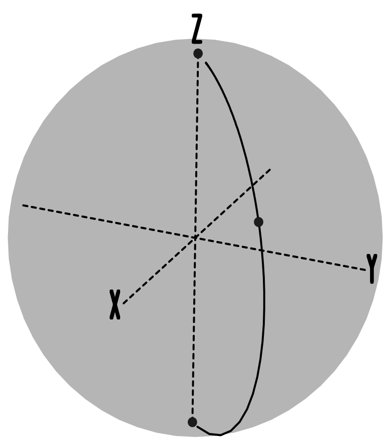
$$y \equiv y(x) = \pm\frac{2}{x}.$$

**Point of Tangency.** Point $P_4 = L_3(\lambda^*)$ is the line $L_3$ point of tangency. With $y = \frac{2}{x}$ and $\lambda^* = \frac{x^2}{x^2+2}$,

$$P_4^+ = L_3(\lambda^*) = ((1-\lambda^*)x, \lambda^* y, 2\lambda^* - 1) = \left(\frac{2}{x^2+2}x, \frac{x^2}{x^2+2}\cdot\frac{2}{x}, 2\frac{x^2}{x^2+2} - 1\right) = \frac{1}{x^2+2}(2x, 2x, x^2-2),$$

and with $y = -\frac{2}{x}$, $P_4^- = \frac{1}{x^2+2}(2x, -2x, x^2-2)$, both for $-\infty < x < \infty$, and both are spherical curves. More specifically, each is half a great circle with $Z$-axis as diameter.

**Locus of Points** $P_4$
(Great semi-circle shown corresponds to $x > 0$ and $y > 0$, in the Plane $Y = X$)




As $P_3 = (x, 0, -1)$, starting at $P_2 = (0, 0, -1)$, moves in the positive direction, $P_5 = (0, y, 1)$, starting "at $\infty$," moves in the negative direction toward $P_1 = (0, 0, 1)$. Point of tangency, $P_4 = \frac{1}{x^2+2}(2x, 2x, x^2-2)$, traverses the great semi-circle shown from $P_2$ to $P_1$.

Three additional great semi-circles correspond to choices $x < 0, y < 0$ (in the $Y = X$ plane), and $x > 0, y < 0$ and $x < 0, y > 0$ (in the $Y = -X$ plane).