Sept 10, 2018

Re: S/O 3 Solution
To Allan Gottlieb, Technology Review Puzzle Corner:

This is a nice problem for 8.01 students at MIT, also nice for some of us who like to review classical mechanics once in a while. It's clear that the football has to be launched with minimum angle θ of arctan (y/x), where there would be infinite velocity, so there must be a singularity there. We also expect a singularity at θ=90 degrees. Somewhere in between, there must be an angle where velocity is minimum.

Pick an angle θ and resolve the velocity v_0 into $v_x = v_0 \cos\theta$ and $v_y = v_0 \sin\theta$. Intercept time t_i for the crossbar (at height y) is $t_i = x/(v_0 \cos\theta)$, which puts a condition for clearing the bar on y position $y(t) = v_y t - \frac{1}{2} g t^2$. We therefore want

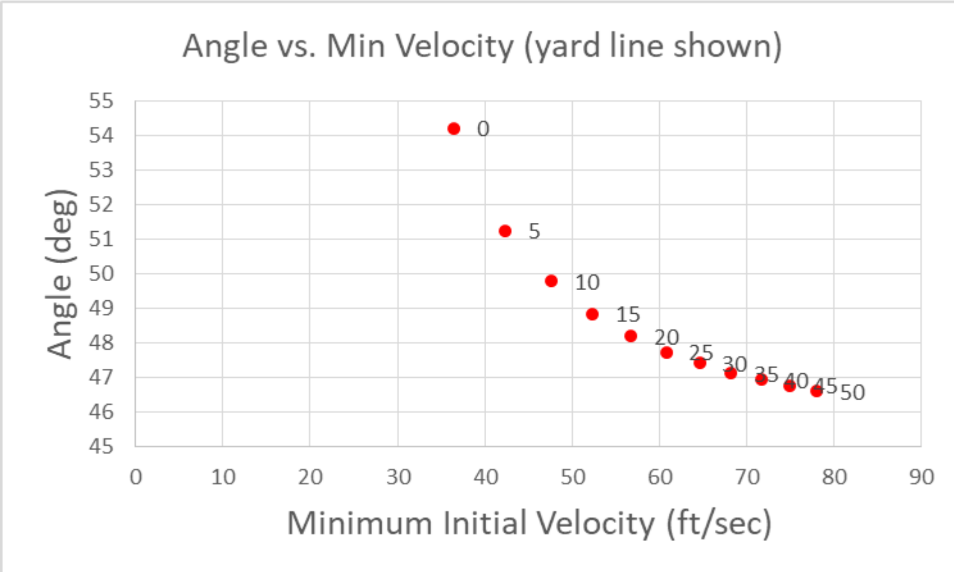
$$y(t_i) = \frac{v_0 x \sin\theta}{v_0 \cos\theta} - \frac{g}{2}\frac{x^2}{v_0^2 \cos\theta^2} = x\tan\theta - \frac{A}{v_0^2 \cos\theta^2} > y$$, where $A=\frac{1}{2}gx^2$. [1]

This means that initial velocity $$v_0^2 > \frac{A/x}{\cos\theta^2(\tan\theta - \tan\theta_{\min})}$$, [2]

where minimum angle θ_{min} = arctan (y/x). This has the expected singularities. To minimize v_0, we maximize the denominator. Setting its derivative equal to zero leads to the condition

$$(\sin 2\theta)\,(\tan\theta - \tan\theta_{\min}) = 1. \quad [3]$$

Now we arrive at my reason for solving the problem iteratively and acquiring the solutions to [2] and [3] with a spreadsheet: I was curious as to what the velocity (ft/sec) and angle (degrees) actually are, under these frictionless conditions, for a football placed on the goal line, 5-yard line, 10-yard line, etc., given that the crossbar is 10 ft. high and 10 yards beyond the goal line (and on the earth's surface, with g=32 ft/sec^2 ☺). A plot is shown below, with yard lines as noted. Note that the optimal angle asymptotically approaches 45 degrees, a result most of us remember from 8.01, the angle for maximum range of a projectile launched on flat ground.



Yours sincerely,

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