Convergence of Means – Technology Review, N/D 2017
Burgess H Rhodes, XVIII, 1960

The Problem N/D 3:

"The arithmetic mean of a_o and b_o is of course $(a_o + b_o)/2$, and the geometric mean is $\sqrt{a_o b_o}$. When $a_o = 2$ and $b_o = 20$, we get 11 and $\sqrt{40} \approx 6.832$. Greg Schaeffer didn't like either answer; he felt a 'reasonable' value would be 'something more like between 8 and 9.' He tried iterating the procedure defining

$$a_{i+1} = (a_i + b_i)/2 \quad \text{and} \quad b_{i+1} = \sqrt{a_i b_i}$$

and the values seem to converge to around 8.5, meeting his intuition. He asks whether both sequences always converge to a common limit, and if so, what is that limit when $a_o = 1$ and b_o is larger?"

Generalization: Consider functions $f(x, y)$ with the property

$$\min\{x, y\} < f(x, y) < \max\{x, y\}, \quad x, y > 0, \quad x \neq y. \tag{P}$$

Such functions produce output values **strictly** between their input values x and y. The **arithmetic mean** $\dfrac{x+y}{2}$, **geometric mean** $\sqrt{xy}$, and **harmonic mean** $\dfrac{2}{1/x + 1/y}$ are functions with this property.

Now consider two functions f and g with property (P). With positive initial values a_o and b_o, define iteration

$$a_{i+1} = f(a_i, b_i) \quad \text{and} \quad b_{i+1} = g(a_i, b_i), \quad i = 0, 1, \ldots \tag{I}$$

Bounds on output values are

$$\min\{a_i, b_i\} < a_{i+1} < \max\{a_i, b_i\} \quad \text{and} \quad \min\{a_i, b_i\} < b_{i+1} < \max\{a_i, b_i\}$$

from which determine

$$\min\{a_i, b_i\} < |a_{i+1} - b_{i+1}| < \max\{a_i, b_i\}.$$

Thus

$$\frac{|a_{i+1} - b_{i+1}|}{|a_i - b_i|} = \rho_i < 1.$$

Consequently,

$$|a_{i+1} - b_{i+1}| = \left(\prod_{j=0}^{i} \rho_j \right) |a_0 - b_0|$$

and because

$$\lim_{i \to \infty} \prod_{j=0}^{i} \rho_j = 0,$$

$$\lim_{i \to \infty} a_{i+1} = \lim_{i \to \infty} b_{i+1} = L.$$

Conclusion: For **any** two functions with property (P) in the rôle of iteration (I), the sequences

$$\{a_i \mid i = 0, 1, \ldots\} \quad \text{and} \quad \{b_i \mid i = 0, 1, \ldots\}$$

have a common limit $L(a_o, b_o)$.

Limiting values $L(a_o, b_o)$ with $a_o = 1$ and the Arithmetic and Geometric Mean Functions
(Note: Convergence to 6 decimals places is fast, no more than 6 iterations for values shown.)

b_o	5	10	15	20	25	50	100	500	1000
$L(1, b_o)$	2.6040	4.2504	5.7499	7.16658	8.5247	14.8223	26.2167	103.3295	189.3883