Convergence of Iterated Means
(Puzzle Corner N/D3 2017, Revisited)
Burgess H. Rhodes, XVIII, 1960

Three Mean Value functions. For a set \( P = \{v_i, \ | \ i = 1, 2, \ldots, n\} \) of positive numbers, the Arithmetic, Geometric, and Harmonic means are defined by

\[
A(v_1, v_2, \ldots, v_n) = \frac{v_1 + v_2 + \cdots + v_n}{n}, \\
G(v_1, v_2, \ldots, v_n) = (v_1 \cdot v_2 \cdots v_n)^{1/n}, \\
H(v_1, v_2, \ldots, v_n) = \frac{n}{1/v_1 + 1/v_2 + \cdots + 1/v_n}.
\]

The harmonic mean is the reciprocal of the average of reciprocals. For all sets \( P \) these means satisfy the Arithmetic/Geometric/Harmonic mean inequality:

\[0 < H(v_1, v_2, \ldots, v_n) \leq G(v_1, v_2, \ldots, v_n) \leq A(v_1, v_2, \ldots, v_n). \quad \text{(AGH)}\]

A Sequence Generated Jointly by Means \( A, G, \) and \( H. \) For a set \( P, \) let \( x_0 = A(v_1, v_2, \ldots, v_n), \ y_0 = G(v_1, v_2, \ldots, v_n), \ z_0 = H(v_1, v_2, \ldots, v_n), \) and let sequence \( \{(x_i, y_i, z_i) \ | \ i = 1, 2, \ldots\} \) be generated by the recursion

\[x_i = A(x_{i-1}, y_{i-1}, z_{i-1}), \quad y_i = G(x_{i-1}, y_{i-1}, z_{i-1}), \quad z_i = H(x_{i-1}, y_{i-1}, z_{i-1}), \quad i = 1, 2, \ldots\]

For \( i = 0, 1, 2, \ldots \), by (AGH),

\[0 < z_i \leq y_i \leq x_i.\]

Define

\[\rho_i = \frac{x_{i+1} - z_{i+1}}{x_i - z_i} = \frac{x_i + y_i + z_i}{3} - \frac{1}{x_i + 1/y_i + 1/z_i} > 0.\]

Then

\[
0 < \rho_i \leq \frac{x_i + x_i + z_i}{3} - \frac{3}{1/x_i + 1/y_i + 1/z_i} = \frac{2x_i + z_i}{3} - \frac{3}{1/x_i + 2/z_i} = \frac{4x_i - z_i}{x_i - z_i} = \frac{4 - z_i/x_i}{6 + 3z_i/x_i} < \frac{2}{3}.
\]

From

\[|x_n - z_n| = \left( \prod_{i=0}^{n-1} \rho_i \right) |x_0 - z_0| \]

Determine

\[
\lim_{n \to \infty} |x_n - z_n| < \lim_{n \to \infty} \left( \frac{2}{3} \right)^n |x_0 - z_0| = 0 \quad \text{(L)}
\]

so that

\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n,
\]

the middle limit following because for all \( i, \ 0 < z_i \leq y_i \leq x_i.\)
**Divergence of an Infinite Product.** The conclusion of the prior example relies on limit (L) for its validity. Suppose \( R = \{ \rho_i \mid i = 1, 2, \ldots \} \) is a sequence of numbers, with \( \rho_i \in (0, 1) \) for all \( i \). Intuitively, one might think

\[
\lim_{n \to \infty} \prod_{i=1}^{n} \rho_i = 0. \tag{C}
\]

However, this limit does not hold for every set \( \mathcal{R} \), although it can hold even when \( \lim_{i \to \infty} \rho_i = 1 \). For example,

\[
\text{with } \rho_i = \frac{i}{i + 1}, \quad \lim_{n \to \infty} \prod_{i=1}^{n} \rho_i = \lim_{n \to \infty} \frac{1}{n + 1} = 0.
\]

**Theorem:** Required conditions for (C) to hold are given* in the following theorem:

If \( a_i < 0 \) or \( a_i > 0 \) for all values of index \( i \) starting with some particular value, then for

\[
\lim_{n \to \infty} \prod_{i=1}^{n} (1 + a_i)
\]

for convergence, it is necessary and sufficient that the series \( \sum_{i=1}^{\infty} a_i \) converges. By convention, an infinite product is said to converge if the limiting value of the partial products in (K) is a non-zero number. Thus (K) will diverge (tend to \( \pm \infty \) or tend to 0) if the series \( \sum_{i=1}^{\infty} a_i \) diverges.

For \( 0 < a_i < 1 \) and \( a_i = 1 + a_i, \ a_i = \rho_i - 1 \) and \(-1 < a_i < 0\). Thus for a set \( \mathcal{R}, a_i < 0 \) for all \( i \), and condition (C) will hold if

\[
\sum_{i=1}^{\infty} a_i = -\infty.
\]

**Counterexample to my “Proof” of a Generalization to Problem N/D3, 2017.** Found in the Overflow Material of Allan Gottlieb’s Puzzle Corner is the following counterexample to my “proof” of a generalization (I thought) of the initial problem N/D3:

“I [Chandler] offer this counterexample: Let \( a_0 = 1 \) and let \( b_0 = 4 \). Define functions \( f \) and \( g \) having the following values for nonnegative integer \( i \):

\[
f(2 - 2^{-i}, 3 + 2^{-i}) = 2 - 2^{-(i+1)}, \quad g(2 - 2^{-i}, 3 + 2^{-i}) = 3 + 2^{-(i+1)}
\]

For other values of \( x \) and \( y \), we only require that the min/max requirements be satisfied for \( f(x, y) \) and \( g(x, y) \).

Then

\[
\lim_{i \to \infty} a_i = 2 \neq \lim_{i \to \infty} b_i = 3.
\]

The “min/max” requirement, satisfied by the three mean value functions above, is

\[
\min\{x, y\} < f(x, y), \ g(x, y) < \max\{x, y\}. \tag{M}
\]

For Chandler’s example and large enough \( i \),

\[
a_i = 1 - \rho_i = 1 - \frac{(3 + 2^{-(i+1)}) - (2 - 2^{-(i+1)})}{(3 + 2^{-i}) - (2 - 2^{-i})} = \frac{1}{2} \cdot \frac{1}{2^{i+1} - 1} < \frac{1}{2},
\]

so that \( \sum_{i=1}^{\infty} a_i \) converges and by the theorem, \( \lim_{n \to \infty} \prod_{i=1}^{n} \rho_i = L \), a non-zero number.

Apparently condition (M) is insufficient to guarantee convergence of sequences generated jointly by \( f \) and \( g \). Symmetry (i.e. \( f(x, y) = f(y, x) \)) is not required. For consider weighted means \( f(x, y) = Mx + My \) and \( g(x, y) = Nx + Ny \) with \( 0 < M, N < 1 \). Convergence occurs if \( 0 < M + N < 2 \). It remains an open challenge to find conditions (replacing (M)) which are sufficient.