Convergence of Iterated Means
(Puzzle Corner N/D3 2017, Revisited)
Burgess H. Rhodes, XVIII, 1960

Three Mean Value functions. For a set $\mathcal{P} = \{v_i, \mid i = 1, 2, \ldots, n\}$ of **positive numbers**, the **Arithmetic**, **Geometric**, and **Harmonic** means are defined by

$$\begin{aligned} A(v_1, v_2, \ldots, v_n) &= \frac{v_1 + v_2 + \cdots + v_n}{n}, \\ G(v_1, v_2, \ldots, v_n) &= (v_1 \cdot v_2 \cdots v_n)^{1/n}, \\ H(v_1, v_2, \ldots, v_n) &= \frac{n}{1/v_1 + 1/v_2 + \cdots + 1/v_n}. \end{aligned}$$

The harmonic mean is the reciprocal of the average of reciprocals. For all sets $\mathcal{P}$ these means satisfy the Arithmetic/Geometric/Harmonic mean inequality:

$$0 < H(v_1, v_2, \ldots, v_n) \le G(v_1, v_2, \ldots, v_n) \le A(v_1, v_2, \ldots, v_n). \qquad \textbf{(AGH)}$$

A Sequence Generated Jointly by Means A, G, and H. For a set $\mathcal{P}$, let $x_0 = A(v_1, v_2, \ldots, v_n)$, $y_0 = G(v_1, v_2, \ldots, v_n)$, $z_0 = H(v_1, v_2, \ldots, v_n)$, and let sequence $\{(x_i, y_i, z_i) \mid i = 1, 2, \ldots\}$ be generated by the recursion

$$x_i = A(x_{i-1}, y_{i-1}, z_{i-1}), \quad y_i = G(x_{i-1}, y_{i-1}, z_{i-1}), \quad z_i = H(x_{i-1}, y_{i-1}, z_{i-1}), \quad i = 1, 2, \ldots$$

For $i = 0, 1, 2, \ldots$, by **(AGH)**,

$$0 < z_i \le y_i \le x_i.$$

Define

$$\rho_i = \frac{x_{i+1} - z_{i+1}}{x_i - z_i} = \frac{\dfrac{x_i + y_i + z_i}{3} - \dfrac{3}{1/x_i + 1/y_i + 1/z_i}}{x_i - z_i} > 0.$$

Then

$$\begin{aligned} 0 < \rho_i &\le \frac{\dfrac{x_i + x_i + z_i}{3} - \dfrac{3}{1/x_i + 1/z_i + 1/z_i}}{x_i - z_i} = \frac{\dfrac{2x_i + z_i}{3} - \dfrac{3}{1/x_i + 2/z_i}}{x_i - z_i} \\ &= \frac{4x_i - z_i}{6x_i + 3z_i} = \frac{4 - z_i/x_i}{6 + 3z_i/x_i} < \frac{2}{3}. \end{aligned}$$

From

$$|x_n - z_n| = \left(\prod_{i=0}^{n-1} \rho_i\right) |x_0 - z_0|$$

Determine

$$\lim_{n\to\infty} |x_n - z_n| < \lim_{n\to\infty} \left(\frac{2}{3}\right)^n |x_0 - z_0| = 0 \qquad \textbf{(L)}$$

so that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n,$$

the middle limit following because for all i, $0 < z_i \le y_i \le x_i$.

Divergence of an Infinite Product. The conclusion of the prior example relies on limit **(L)** for its validity. Suppose $\mathcal{R} = \{\rho_i \mid i = 1, 2, \ldots\}$ is a sequence of numbers, with $\rho_i \in (0,1)$ for all i. Intuitively, one might think

$$\lim_{n\to\infty} \prod_{i=1}^{n} \rho_i = 0. \tag{C}$$

However, this limit **does not** hold for every set $\mathcal{R}$, although it can hold even when $\lim_{i\to\infty} \rho_i = 1$. For example,

$$\text{with } \rho_i = \frac{i}{i+1}, \quad \lim_{n\to\infty} \prod_{i=1}^{n} \rho_i = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Theorem: Required conditions for **(C)** to hold are given* in the following theorem:

If $a_i < 0$ or $a_i > 0$ for all values of index i starting with some particular value, then for

$$\lim_{n\to\infty} \prod_{i=1}^{n} (1 + a_i) \tag{K}$$

to converge, it is necessary and sufficient that the series $\sum_{i=1}^{\infty} a_i$ converges. **By convention**, an infinite product is said to converge if the limiting value of the partial products in **(K)** is a **non-zero** number. Thus **(K)** will **diverge** (tend to $\pm\infty$ or tend to 0) if the series $\sum_{i=1}^{\infty} a_i$ diverges.

For $0 < \rho_i < 1$ and $\rho_i = 1 + a_i$, $a_i = \rho_i - 1$ and $-1 < a_i < 0$. Thus for a set $\mathcal{R}$, $a_i < 0$ for all i, and condition **(C)** will hold if

$$\sum_{i=1}^{\infty} a_i = -\infty.$$

Counterexample to my "Proof" of a Generalization to Problem N/D3, 2017. Found in the Overflow Material of Allan Gottlieb's Puzzle Corner is the following counterexample to my "proof" of a generalization (I thought) of the initial problem N/D3:

> "I [Chandler] offer this counterexample: Let $a_0 = 1$ and let $b_0 = 4$. Define functions f and g having the following values for nonnegative integer i:
>
> $$f(2 - 2^{-i}, 3 + 2^{-i}) = 2 - 2^{-(i+1)}; \;\; g(2 - 2^{-i}, 3 + 2^{-i}) = 3 + 2^{-(i+1)}$$
>
> For other values of x and y, we only require that the min/max requirements be satisfied for $f(x,y)$ and $g(x,y)$.
>
> "Then
>
> $$\lim_{i\to\infty} a_i = 2 \neq \lim_{i\to\infty} b_i = 3."$$

The "min/max" requirement, satisfied by the three mean value functions above, is

$$\min\{x,y\} < f(x,y), g(x,y) < \max\{x,y\}. \tag{M}$$

For Chandler's example and large enough i,

$$a_i = 1 - \rho_i = 1 - \frac{(3 + 2^{-(i+1)}) - (2 - 2^{-(i+1)})}{(3 + 2^{-i}) - (2 - 2^{-i})} = \frac{1}{2} \cdot \frac{1}{2^{i-1} - 1} < \frac{1}{2},$$

so that $\sum_{i=1}^{\infty} a_i$ converges and by the theorem, $\lim_{n\to\infty} \prod_{i=1}^{n} \rho_i = L$, a non-zero number.

Apparently condition **(M)** is insufficient to guarantee convergence of sequences generated jointly by f and g. Symmetry (*i.e.* $f(x,y) = f(y,x)$) is not required. For consider weighted means $f(x,y) = Mx + \bar{M}y$ and $g(x,y) = \bar{N}x + Ny$ with $0 < M, N < 1$. Convergence occurs if $0 < M + N < 2$. It remains an open challenge to find conditions (replacing **(M)**) which are sufficient.

* Cited as 0.252 in Gradshteyn & Ryzhik, *Tables of Integrals, Series, and Products*, Seventh Edition, Elsevier Inc., ©2007.