Problem J/F 2 from Puzzle Corner – Technology Review, 2017
Burgess H Rhodes, XVIII, 1960

The Problem:

"Richard Bronowitz offers this radar detection problem. Assume that a radar has a detection threshold requiring at least nine successful pulse returns out of 10 successive pulses. Furthermore, once an object is is detected, it remains detected – i.e., there are no lost contacts. The probability that a pulse is successfully detected is p, and pulse results are independent. What is the probability of detection given N pulses?"

Overview. This problem may be re-characterized and generalized:

Flip a biased coin with $\Pr\{\text{head}\} = p$ until attainment of the **goal** "K heads occur among the final M flips for the first time at the N^{th} flip". Determine the cumulative probability distribution for random variable N.

A Solution. This situation lends itself to modeling as an **Absorbing Markov Chain***. Definitions of states and transitions are:

State: Strings of length M of 0's and 1's, a 1 representing a head flip. Such strings containing K or more 1's are aggregated to constitute the one absorbing state G, representing the attainment of the goal. All other strings are individual transient states.

Transition: A flip of the coin. If $b_1 b_2 \dots b_M$ is a current transient state (b_i is a binary digit), then transition probabilities are

$$\begin{aligned}\Pr\{b_2 \dots b_M 1 \mid b_1 b_2 \dots b_M\} &= p,\\ \Pr\{b_2 \dots b_M 0 \mid b_1 b_2 \dots b_M\} &= 1 - p = q, \quad \text{and}\\ \Pr\{G \mid G\} &= 1.\end{aligned}$$

Very Small Example. With parameter values $K = 2$ and $M = 3$, the one-step transition matrix T is

$$T = \begin{array}{c} \\ G \\ 000 \\ 001 \\ 010 \\ 100 \end{array} \begin{array}{c} \begin{array}{ccccc} G & 000 & 001 & 010 & 100 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & q & p & 0 & 0 \\ p & 0 & 0 & q & 0 \\ p & 0 & 0 & 0 & q \\ 0 & q & p & 0 & 0 \end{pmatrix} \end{array}$$

Define state vector S_n, the condition of the process at time n (after n flips), by

$$S_n = (\, G(n), \quad s_n(1), \quad s_n(2), \quad s_n(3), \quad s_n(4)\,),$$

in which $G(n) = \Pr\{N \le n\}$ and $s_n(i) = \Pr\{\text{at time } n \text{ the process is in transient state } i\}$.

With initial state vector $S_0 = (0, 1, 0, 0, 0, 0)$, successive state vectors are determined by recursion:

$$S_n = S_{n-1} T, \quad n = 1, 2, \dots \qquad \textbf{(R)}$$

The sequence

$$\{G(n) \mid n = 0, 1, 2, \dots\}$$

is the cumulative probability distribution for random variable N.

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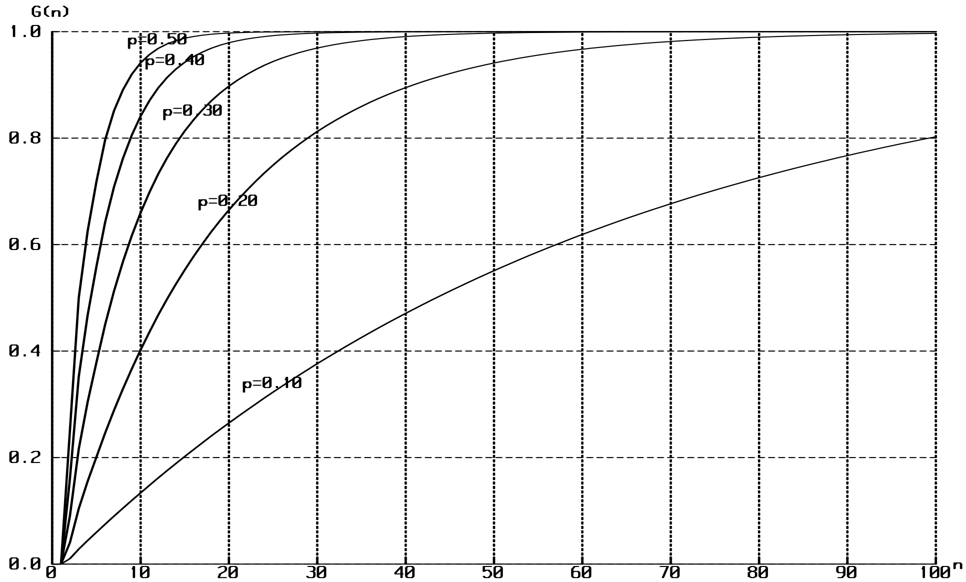
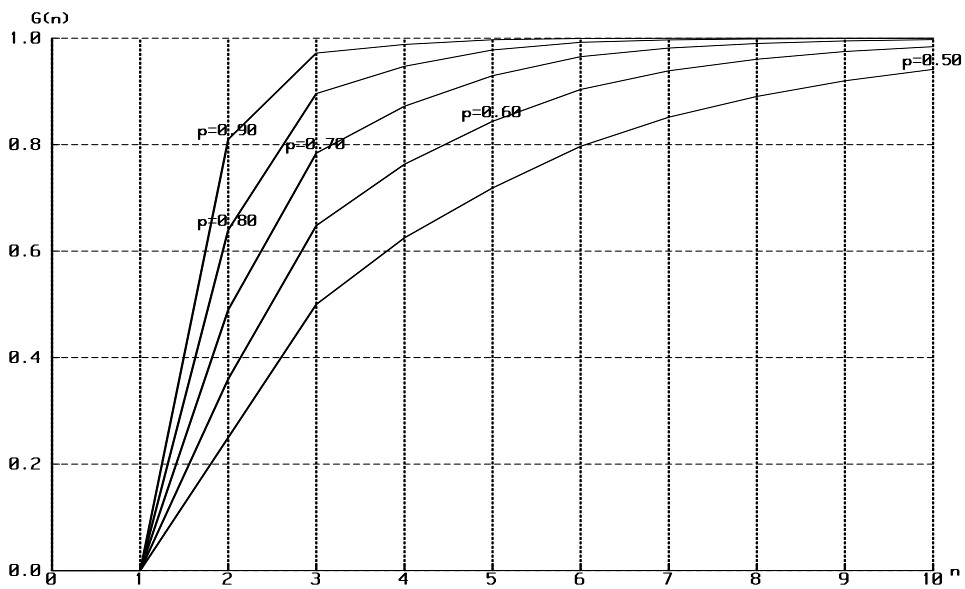
* See, for example, the chapter on Markov Chains in *Introduction to Operations Research* by Hillier & Lieberman, McGraw Hill, any edition.

With a CAS, symbolic values for $G(n)$ can be obtained. Numerical values for $G(n)$ can be obtained for a desired p. Graphs of $G(n)$ for $p = 0.1$ to $p = 0.9$ are given for the **Very Small Example**:





Large Values for Parameters. For large values of K and M the approach above is impractical. In general, matrix T would have $2^M - \sum_{\ell=K}^{M-1} \binom{M}{\ell}$ rows and columns. For values $K = 9$ and $M = 10$ in the original problem statement, the size of matrix T is 1014×1014. Because matrix T is sparse, with just two non-zero entries per row, an alternative is to replace the recursion (**R**) with explicit equations. For the **Very Small Example** and $n = 1, 2, \ldots$, these equations are

$$\begin{aligned}
G(n) &= G(n-1) + p \cdot [s_{n-1}(2) + s_{n-1}(3)],\\
s_n(1) &= q \cdot [s_{n-1}(1) + s_{n-1}(4)],\\
s_n(2) &= p \cdot [s_{n-1}(1) + s_{n-1}(4)],\\
s_n(3) &= q \cdot s_{n-1}(2), \quad \text{and}\\
s_n(4) &= q \cdot s_{n-1}(3).
\end{aligned}$$

As the coin is flipped the Markov Chain transitions from state to state. Denote the transient states by $B_1, B_2, \ldots$ The possible transitions are depicted here:

State Transition Diagrams (Transient States)

Next Flip a Tail (q)

$0b_1 \ldots b_{M-1}\ [= B_{i_0}]$ $\searrow$

$b_1 b_2 \ldots b_{M-1} 0\ [= B_{j_0}]$

$1b_1 \ldots b_{M-1}\ [= B_{i_1}]$ $\nearrow$

Next Flip a Head (p)

$0b_1 \ldots b_{M-1}\ [= B_{i_0}]$ $\searrow$

$b_1 b_2 \ldots b_{M-1} 1\ [= B_{j_1}]$

$1b_1 \ldots b_{M-1}\ [= B_{i_1}]$ $\nearrow$

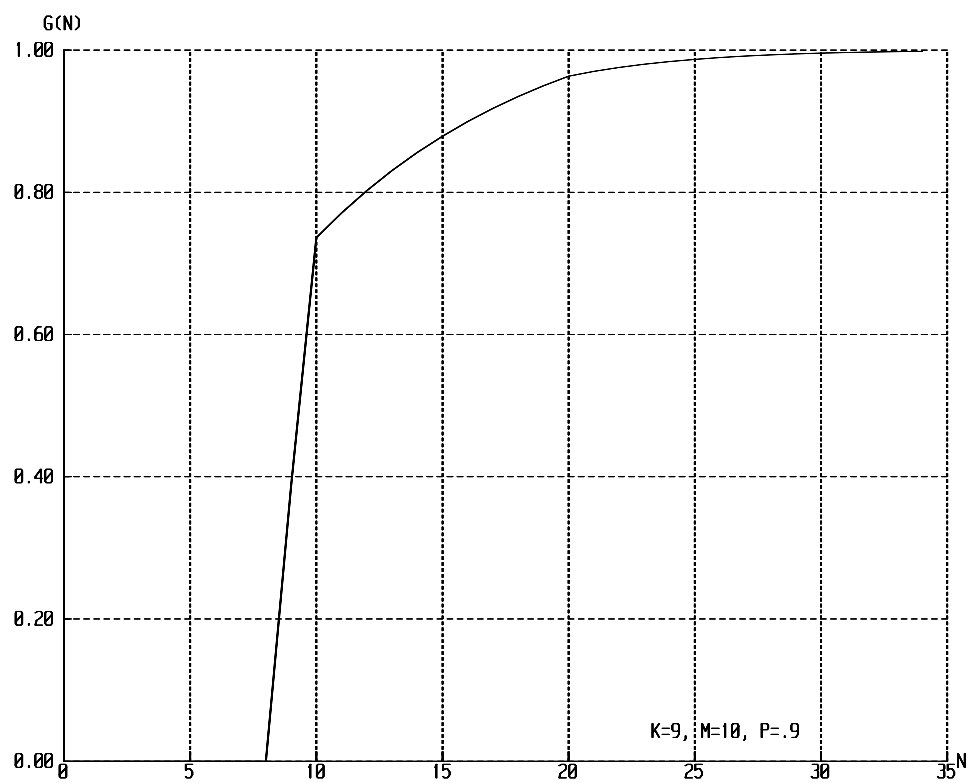
Those states which contain K (or more) 1's are aggregated into a single absorbing state G. If state B_{j_1} contains K 1's, then $B_{j_1} \equiv G$. In general, each state has two states preceeding.

As earlier, $s_n(i)$ is the probabilty that the system is in transient state B_i at time n. Then equations replacing recursion (**R**) are of the form:

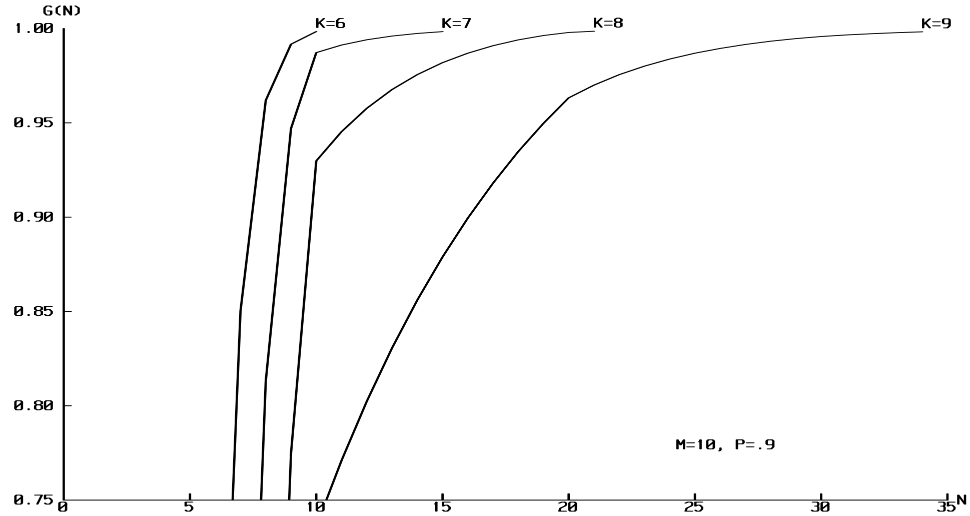
$$s_n(j_0) = q \cdot [s_{n-1}(i_0) + s_{n-1}(i_1)],$$
$$s_n(j_1) = p \cdot [s_{n-1}(i_0) + s_{n-1}(i_1)].$$

Numerical Examples. Values for $G(n)$, $n = 0, 1, \ldots$, including initial parameter values $K = 9$ and $M = 10$, are given in the following figures. Variations in the three parameters K, M, and p are explored. In each plot, value n extends up the first flip on which $G(n) \geq 0.998$ (with a maximum 200 flips). For emphasis, figures displaying parameter variations have origins at $(0, 0.75)$.

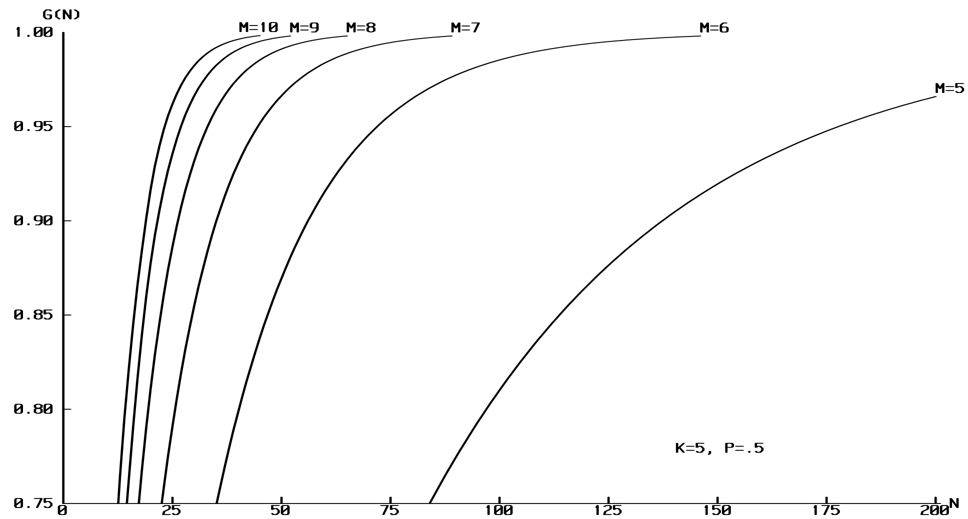
Cumulative Probability Distribution $G(n)$ with $K = 9$ and $M = 10$ ($p = 0.9$)



Cumulative Probability Distributions $G(n)$ with $M = 10$ and $p = 0.9$ ($K = 6[1]9$)



Cumulative Probability Distributions $G(n)$ with $K = 5$ and $p = 0.5$ ($M = 5[1]10$)



Cumulative Probability Distribution $G(n)$ with $K = 9$ and $M = 10$ ($p = 0.7[.05].95$)