M/J2 Solution: A = 6, B = 7

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This puzzle is similar in spirit to the famous Cheryl's Birthday problem that went viral on social media in 2015. And like the previous puzzle, this one may be solved by listing all candidate solutions in a table, and then by processing each statement in turn and eliminating rows and columns until only one solution remains, which must be the answer.

Let's name each statement in order:

> S1: A says, "There is no way you can know the number on your hat."
> S2: B says, "I don't know my number."
> S3: A says, "I don't know my number."
> S4: B says, "I now know my number."

To save time, I construct the first few rows and columns of the table ***after*** processing statement S1, as shown below. Each row shows a possible value for A, while each column a possible value for B. Each non-empty cell shows a pair (*x*,*y*) where A = *xy* and B = *x*+*y*.

A \ B	5	7	9	10	11	13	15	16	17	19
4	1,4									
6	2,3	1,6								
8			1,8							
9				1,9						
10		2,5			1,10					
12		3,4				1,12				
14			2,7				1,14			
15								1,15		
16				2,8					1,16	
18			3,6		2,9					1,18

Statement S1 means that *xy* cannot be a prime number, because if it is, then x=1, and B would immediately know his number is 1+y. But S1 also says something stronger: the fact that B cannot possibly know his own number must mean that what A sees cannot be a prime plus 1. For example, if A sees 8, although there are many ways for two numbers to sum to 8 (3+5, 1+7, 2+6, etc), one of these is 1+7, which leaves the possibility that B could know his number (if he saw 7).

In the table above, the rows show composite (non-prime) numbers, while the columns show only numbers that can never be written as 1 + some prime. Now let's process each of the remaining statements in turn.

S2 says B is ignorant about his own number. Note that B sees A’s number, i.e. one of the rows in the table. If a particular row has only 1 entry, then B would immediately know his own number. Since this is not the case, we can eliminate the rows A=4, A=8, A=9 and A=15, as these have only 1 non-empty entry.

A \ B	5	7	9	10	11	13	15	16	17	19
~~4~~	~~1,4~~									
6	2,3	1,6								
~~8~~			~~1,8~~							
~~9~~				~~1,9~~						
10		2,5			1,10					
12		3,4				1,12				
14			2,7				1,14			
~~15~~								~~1,15~~		
16				2,8					1,16	
18			3,6		2,9					1,18

Note that A is following this reasoning too, and that the number A sees is one of the columns in the table. Thus, when A says S3, he must be seeing a column with multiple non-empty entries. Hence we can eliminate column B=5, which has a single entry. Note that for all columns 13 and above, there will be at least one other entry in these columns in some rows farther down the table. For instance, for B=13, another possible sum is 6+7, whose product is 42 (which is not a prime), and will thus appear in row A=42. The table now becomes:

A \ B	~~5~~	7	9	10	11	13	15	16	17	19
~~4~~	~~1,4~~									
6	~~2,3~~	1,6								
~~8~~			~~1,8~~							
~~9~~				~~1,9~~						
10		2,5			1,10					
12		3,4				1,12				
14			2,7				1,14			
~~15~~								~~1,15~~		
16				2,8					1,16	
18			3,6		2,9					1,18

Finally, let’s process statement S4. This must mean that B sees a row with only a single entry, i.e. the row A=6. Hence the answer.