Problem M/A2 from Puzzle Corner – Technology Review, 2016
Burgess H Rhodes, XVIII, 1960

The Problem:

> "Brian Cook likes to think big. He wished to merge N companies into one giant company and asks: If only one merger can be performed at a time and only pairwise mergers are allowed ($2 \to 1$), how many distinct ways are there to form the giant company? For example, if $N = 3$, then there are only three possible orders in which to merge them: ((12)3), ((13)2), and (1,(23)). For extra credit, try removing the restriction to pairwaise mergers."

Pairwise Mergers. Let $\mathcal{C}_n$ denote a set of n objects to be paired. A pairwise merger is a transformation

$$\mathcal{C}_n \to \mathcal{C}_{n-1}$$

which can occur in $\binom{n}{2}$ ways. The giant company is the result of

$$\mathcal{C}_N \to \mathcal{C}_{N-1} \to \cdots \to \mathcal{C}_2 \to \mathcal{C}_1$$

which can occur in

$$F_2(N) = \prod_{k=0}^{N-2} \binom{N-k}{2}$$

distinct ways. When the merger process reaches state $\mathcal{C}_2$, the final merger ($k = N-2$) is automatic, leaving the number of ways unchanged. Thus

$$F_2(N) = \prod_{k=0}^{N-3} \binom{N-k}{2}.$$

Mergers by Subsets of Size M. For $n > M$, an M-wise merger is a transformation

$$\mathcal{C}_n \to \mathcal{C}_{n-(M-1)}$$

which can occur in

$$\binom{n}{M}$$

ways. If $n \le M$ the merger $\mathcal{C}_n \to \mathcal{C}_1$ is automatic, without changing the number of distinct ways that the giant company can be formed. Thus from

$$N - k(M-1) \ge M \quad \text{implies} \quad k \le \left\lfloor \frac{N-M}{M-1} \right\rfloor = K_{\max}$$

determine

$$F_M(N) = \prod_{k=0}^{K_{\max}} \binom{N-k(M-1)}{M}.$$

A Few Values. For $N = 8$,

$$F_2(8) = 1{,}587{,}600, \quad F_3(8) = 4{,}480, \quad \text{and} \quad F_4(8) = 350.$$