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J/A 3: (Restated) For any triangle, find the relationship between in-radius r, circumradius R and the distance X between the in-center and circumcenter.

The answer is $X^2 = R^2 - 2Rr$; one example is illustrated in Figure 1. Note that there exists no triangle with $r > R/2$ (equality happens for an equilateral triangle), so X is always real.

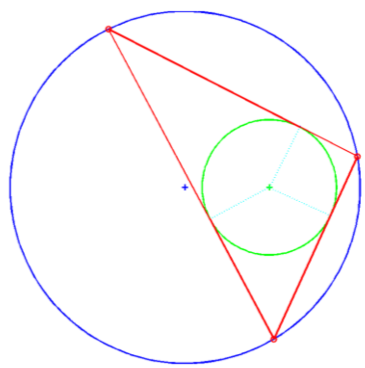

Figure 1: Case $X^2 = R^2 - 2Rr$

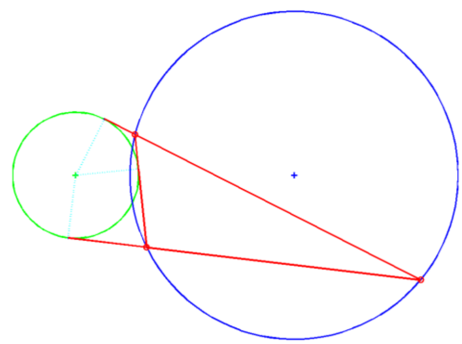

Figure 2: Case $X^2 = R^2 + 2Rr$

In a liberal interpretation of the problem, there is a second solution, namely $X^2 = R^2 + 2Rr$. This occurs when we allow the "inscribed circle" to lie outside the triangle but, like the true inscribed circle, we require it to be tangent to all three sides of the triangle, allowing two of the sides to extend, as shown in Figure 2.

The relationship can be discovered numerically by generating random triangles, calculating the inscribing and circumscribing circles, then plotting for each random trial a point at $(x, y) = (X/R\,, r/R)$. The parabola $x^2 = 1 - 2y$ is then immediately apparent, and it checks numerically to the precision of the arithmetic employed. This reveals the result with near certainty, but such a demonstration falls short of rigorous proof.

To derive the result rigorously, we turn to an algebraic formulation. Figure 3 shows the notation: A, B, C are the vertices of the triangle, P, Q are the centers of its inscribed circle and circumscribed circle, resp., D, E, F are the points of tangency of the inscribed circle. We will use a complex vector formulation, that is, a point (x, y) in Cartesian coordinates will be considered as the point $x + iy$ in the complex plane of our diagram. Let P be the origin, i.e., $P = 0$, and place the circumcenter Q on the negative real axis at $Q = -X$.

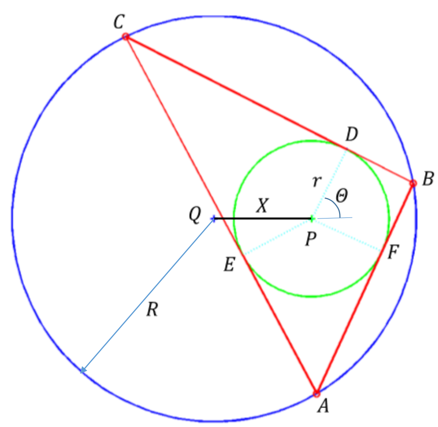

Figure 3. Notation

Suppose that D is at angle Θ relative to P, so it is the complex point $D = re^{i\Theta}$. To make all derivations easier, we introduce $\theta := e^{i\Theta}, \theta' := e^{-i\Theta}$, so $D = r\theta$. (x' will mean the complex conjugate of x throughout.) Turning by 90° in the complex plane is equivalent to multiplication by i, so for some real number λ, point C can be written as $C = D + \lambda i\theta = (r + i\lambda)\theta$. But C is distance R from circumcenter Q, so we must have

$$(C + X)(C' + X) = R^2,$$

where we have used $X' = X$, because X is real. Substitution for C gives

$$((r + \lambda i)\theta + X)((r - \lambda i)\theta' + X) = R^2,$$

which expanded and rearranged becomes

$$\lambda^2 + iX(\theta - \theta')\lambda + X^2 + r^2 - R^2 + Xr(\theta + \theta') = 0.$$

This is a quadratic in λ and since point B is defined by the same conditions as point C, the two roots of the quadratic, say λ_1, λ_2, give points C and B as

$$C = (r + i\lambda_1)\theta, \qquad B = (r + i\lambda_2)\theta.$$

Since $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$, the quadratic equation above implies that

$$(\lambda_1 + \lambda_2) = -iX(\theta - \theta'), \qquad \lambda_1\lambda_2 = X^2 + r^2 - R^2 + Xr(\theta + \theta').$$

Now, since the in-center lies on the angle bisectors, point E is the reflection of D across line $\overleftrightarrow{PC}$ and point F is the reflection of D across line $\overleftrightarrow{PB}$. A little complex plane geometry shows that this implies

$$E = \frac{C}{C'}D' = \frac{(r + i\lambda_1)r}{(r - i\lambda_1)}, \qquad F = \frac{B}{B'}D' = \frac{(r + i\lambda_2)r}{(r - i\lambda_2)},$$

where we used the fact that $\theta\theta' = 1$. Finally, A lies at the intersection of lines $\overleftrightarrow{CE}$ and $\overleftrightarrow{BF}$. Using the expressions for B, C, E, F, some messy algebra gives A as

$$A = \frac{(r + i\lambda_1)(r + i\lambda_2)r\theta}{r^2 + \lambda_1\lambda_2}.$$

But we require that A also lies on the circumcircle, so

$$(A + X)(A' + X) = R^2.$$

Expanding this and clearing denominators gives a polynomial in $R, r, X, \theta, \theta', \lambda_1, \lambda_2$. That polynomial has a factor $\lambda_1\lambda_2$ that can be discarded, since it corresponds to a degenerate situation in which one side of the triangle has length zero. The remaining polynomial can be rearranged so that λ_1 and λ_2 only appear in the combinations $(\lambda_1 + \lambda_2)$ or $\lambda_1\lambda_2$, both of which we wrote expressions for above. Substituting these expressions to eliminate λ_1 and λ_2 and simplifying the result --- a messy operation for which I used a computer algebra program --- gives a polynomial in which θ and θ' drop out, using $\theta\theta' = 1$. This expression factors as

$$(X^2 - R^2 + 2Rr)(X^2 - R^2 - 2Rr) = 0.$$

The two factors are exactly the ones illustrated in Figures 1 and 2.

By the way, the fact that angle Θ drops out of the final expression implies a surprising fact: if R, r, X satisfy the relationship, then there are an infinite number of triangles that can be simultaneously inscribed/circumscribed in the same circles. We can choose any angle Θ and complete the construction using the formulas for A, B, C above. Figure 4 illustrates this for the circles from Figure 1.

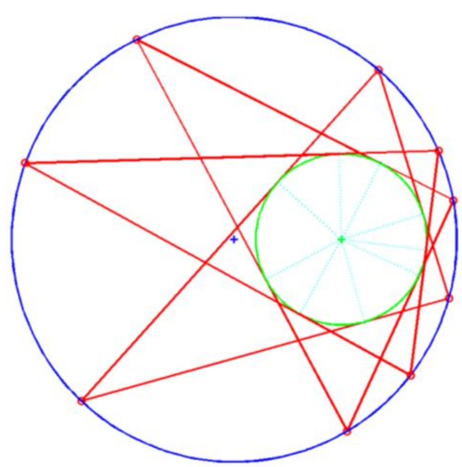

Figure 4. Three triangles inscribed/circumscribed by the same circles