July 6, 2016

Re: J/A 3 Solution, Rev. 0
To Allan Gottlieb, Technology Review Puzzle Corner:

Find a relationship among R, r, and X for a triangle with circumscribed circle (radius R), inscribed circle (radius r), and offset X between the centers.

I think this problem is best treated starting with the unit circle as the circumscribed circle, where the triangle is defined by three angles θ_1, θ_2, and θ_3. For the moment, R=1, and in the end everything (i.e., r and X) will scale by the chosen R, angles invariant.

The mathematical literature on triangles with circumscribed and inscribed circles like this is nicely summarized in this useful 2-page article: http://www.kurtnalty.com/Triangle. Here are some basic definitions from that article:

Define three points by $\vec{P1}$, $\vec{P2}$, and $\vec{P3}$. From these, define the directed line segments and lengths:

$$\begin{aligned}\vec{A} &= \vec{P2} - \vec{P1}\\ \vec{B} &= \vec{P3} - \vec{P2}\\ \vec{C} &= \vec{P1} - \vec{P3}\end{aligned}$$

Notice these are defined cyclically such that $\vec{A} + \vec{B} + \vec{C} = 0$.
Calculate lengths, semiperimeter and area

$$\begin{aligned}a &= |\vec{A}|\\ b &= |\vec{B}|\\ c &= |\vec{C}|\\ s &= (a+b+c)/2\\ K &= \sqrt{s*(s-a)*(s-b)*(s-c)}\end{aligned}$$

***

To get X, we're interested in the difference between the incenter vector **I** and the circumcenter vector **E**. We're also interested in how R and r relate to a, b, c, and derived quantities s and K as above. Again, from Nalty's article:

Incenter

The inscribed circle center is given by the weighted average of the coordinates by the opposite side. The center is the intersection of the three angle bisectors.

$$\vec{I} = \frac{(b * \vec{P}1 + c * \vec{P}2 + a * \vec{P}3)}{a + b + c} \tag{2}$$

The radius of the inscribed circle is

$$\begin{aligned} r &= \sqrt{(s-a) * (s-b) * (s-c)/s} \\ &= K/s \end{aligned}$$

Circumcenter or Exocenter

The circumcenter or exocenter is found by erect perpendicular bisectors from each side. This center is the intersection of these lengths. Building off the $\vec{P}1$ - $\vec{P}3$ line, we have

$$\vec{E} = (1/2)(\vec{P}1 + \vec{P}3) + \frac{\vec{A} \cdot \vec{B}}{8K^2}(\vec{C} \times (\vec{A} \times \vec{B})) \tag{3}$$

The radius of the circumscribed circle is

$$R = \frac{abc}{4K}$$

***

Given Eq. (3) for **E**, it looks like it was a good idea to locate **E** at the origin and not have to wrestle with vector products, unless we want to use **E**=**0** somehow. Now X=|**I**|.

On the unit circle, $\vec{P}_1. = (\cos\theta_1, \sin\theta_1)$, etc. When the magnitudes a, b, and c are computed from the definitions above, the trigonometric angle difference formula and half-angle formula give the following (define $\theta_{21}=\theta_2-\theta_1$ and similarly):

$$a = \sqrt{2(1-\cos(\theta_{21})} = 2\sin\left(\frac{\theta_{21}}{2}\right)$$

$$b = \sqrt{2(1-\cos(\theta_{32})} = 2\sin\left(\frac{\theta_{32}}{2}\right)$$

$$c = \sqrt{2(1-\cos(\theta_{13})} = 2\sin\left(\frac{\theta_{13}}{2}\right).$$

From Nalty's article we already know how r and R relate to a, b, and c. Now it remains to find X as the magnitude of **I**, again using the angle difference formula:

$$X = \frac{\left|b\vec{P}_1 + c\vec{P}_2 + a\vec{P}_3\right|}{a+b+c} = \frac{\left|(b\cos\theta_1 + c\cos\theta_2 + a\cos\theta_3, b\sin\theta_1 + c\sin\theta_2 + a\sin\theta_3)\right|}{a+b+c}$$

$$= \frac{\sqrt{a^2 + b^2 + c^2 + 2cb\cos(\theta_{21}) + 2ab\cos(\theta_{13}) + 2ac\cos(\theta_{32})}}{a+b+c}.$$

Now we use expressions from above such as $2\cos\theta_{21}=2-a^2$ to get

$X = \frac{\sqrt{(a+b+c)^2 - abc(a+b+c)}}{a+b+c} = \sqrt{1-\frac{abc}{a+b+c}}$. But from Nalty, above, abc=4K and r=2K/(a+b+c), so

$X = \sqrt{1-2r}$. This is the unit circle solution, so for any R, $X = R\sqrt{1-2\frac{r}{R}}$. This checks out for the equilateral triangle, where on the unit circle R=1, r=1/2, and the centers of the two circles are well known to coincide at the origin, giving X=0. I think this is the "relationship" among r, R, and X that is being sought. I believe the equations above are also sufficient to solve for a, b, and c (and thus the relative angle placements, defining the triangle) given a compatible set of r, R, and X numbers.

Yours sincerely,

Timothy J. Maloney '71
Palo Alto, CA