PROPOSED SOLUTION TO PROBLEM J/F 3, PUZZLE CORNER, MIT NEWS 2015

We are asked to optimize the constant bet fraction c of stake S during a game of repeated coin flips, where a win yields a profit equal to the bet and a loss costs a fraction f of the bet.

Several years ago (*Tech Review 113: M63, 2010*) my father and I teamed up to solve a problem of pairs, calculating the odds of winning a tennis game that can end after deuce only when an odd-even pair of consecutive points is won. Now we've paired up again to propose a solution to this problem of coin-flip pairs.

Each winning coin flip increments the current stake by the factor $(1 + c)$, whereas a losing coin flip reduces the stake by the factor $(1 - fc)$. So, for example, starting with stake S_0, after six coin flips the stake might be:

$$S = S_0(1 - fc)(1 - fc)(1 + c)(1 - fc)(1 + c)(1 + c)$$

Assuming equal numbers of wins and losses over many flips and pairing them, the stake after *n* pairs is:

$$S_n = S_0[(1 + c)(1 - fc)]^n \qquad (1)$$

We wish to maximize the fractional change in stake for each win-loss pair, which equals $(1 + c)(1 - fc) - 1$ or:

$$\frac{dS}{S} = (1 - f)c - fc^2 \qquad (2)$$

which is a simple quadratic equation, a concave-downward parabola whose vertex is the optimal bet fraction, c^*. The vertex is determined by setting the first derivative of equation (2) equal to zero, yielding the relation between c^* and f:

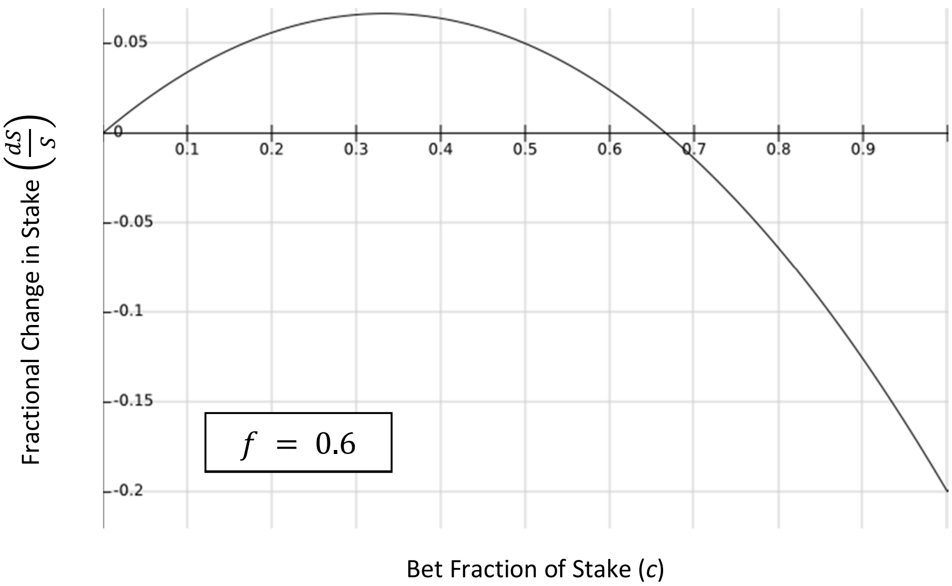
$$c^* = \frac{1-f}{2f} \qquad (3)$$

Note that c^* is independent of S, making a constant bet fraction strategy optimal. Substituting c^* into equation (2) yields, after rearrangement, the best increment in stake possible per win-loss pair for a given f:

$$\frac{(dS)_{max}}{S} = \frac{(1-f)^2}{4f} \qquad (4)$$

Note that a positive increment exists for all f between 0 and 1, so in this range the player will always profit in time if c^* is employed.

The problem asks specifically about the case where $f = 0.6$. Substituting into equation (2) yields the following relation:



In the case of $f = 0.6$, chosen values of c between 0 and $\frac{2}{3}$ result in a net gain over time. The optimal bet fraction (c^*) [by equation (3)] is $\frac{1}{3}$, yielding a $6.\overline{6}\%$ gain in stake per win-loss pair [by equation (4)] and a doubling of the stake in 22 coin flips [by equation (1)]. A progressively larger loss in stake occurs as c increases above $\frac{2}{3}$. If $c = 1$, the worst possible strategy, 20% of the stake is lost with each win-loss pair and, after 22 coin flips, only 8.6% of the original stake remains.

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