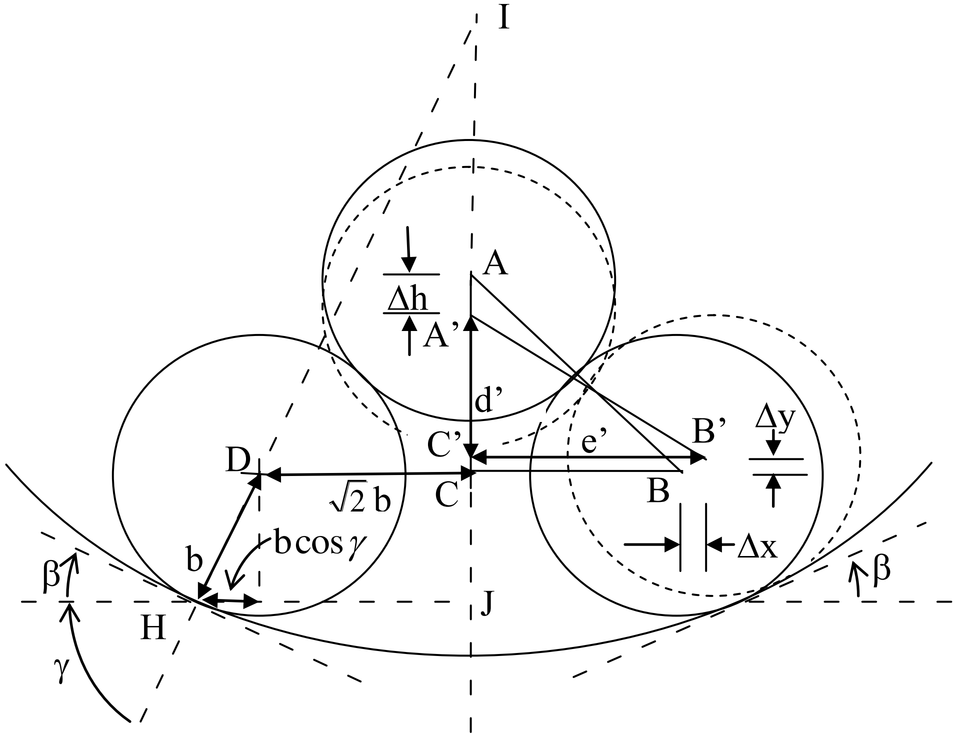
Four balls of radius b sit in a square pattern at the bottom of a bowl of radius a. A fifth ball, the upper ball, is placed on top. Find the condition of stability.




A plane that includes the centers of the upper ball and two diagonally opposite lower balls is shown in the diagram. Point A is the center of the upper ball, Points D and B are the centers of the left and right lower balls. The centers of all four lower balls (two of which are not in the plane of the diagram) form a square with sides of length $2b$. The length |DB| is $2\sqrt{2}b$, since this is the distance between diagonally opposite corners of the square. Point C is located midway between D and B, thus |DC| = |CB|= $\sqrt{2}b$. In triangle ABC, define the lengths of the sides as $d$ = |AC| and $e$ = |CB|. We already know that $e = \sqrt{2}b$, and we know the hypotenuse |AB| is the line between the centers of two balls in contact, with a length $2b$. The Pythagorean theorem tells us the length of the remaining side $d = \sqrt{2}b$. Imagine now that the upper ball descends a distance $\Delta h$, such that Point A moves to a new location A'. The lower balls must all move in order to accommodate this. The result is a new triangle A'B'C'. Point B moves horizontally a distance $\Delta x$ and vertically a distance $\Delta y$ to Point B'. Point C must rise by the same amount $\Delta y$. Let $\Delta d$ be the amount by which $d$ shortens. The lengths of the new sides are

$$d' = |A'C'| = d - \Delta d = \sqrt{2}b - \Delta d$$
$$e' = |B'C'| = e + \Delta x = \sqrt{2}b + \Delta x$$

The balls are still in contact, so the hypotenuse |A'B'| remains fixed at $2b$. The Pythagorean theorem applied to A'B'C' gives

$$(d-\Delta d)^2+(e+\Delta x)^2=4b^2$$

Multiplying this out and neglecting products of small quantities gives $\Delta d = \Delta x$. The amount by which the ball descends is the net result of the drop due to $\Delta d$ and the rise due to $\Delta y$:

$$\Delta h=\Delta d-\Delta y=\Delta x-\Delta y$$

In order for this motion to happen spontaneously, the gravitational energy released from the upper ball descending must be at least as great as the energy required to make the four lower balls rise:

$$\Delta h >= 4\Delta y$$

We take the equal sign in this expression to define the boundary of stability. Eliminating $\Delta h$ from these last two expressions gives

$$4\Delta y=\Delta x-\Delta y$$

From this we obtain

$$\frac{\Delta y}{\Delta x}=\frac{1}{5}=\tan\beta$$

where $\beta$ is the slope of the bowl at the contact point. Now consider triangle HIJ, in which Point I is the center of circle defining the surface of the bowl, and the length |HI| is the radius $a$. The angle IHJ is the complement of $\beta$, defined as $\gamma=\pi/2-\beta$, which is given by

$$\gamma=\tan^{-1}5$$

Angle IHJ also has the following relation:

$$\cos\gamma=\frac{|\mathrm{HJ}|}{|\mathrm{HI}|}=\frac{b\cos\gamma+b\sqrt{2}}{a}$$

But

$$\cos\gamma=\cos(\tan^{-1}5)=\frac{1}{\sqrt{5^2+1}}$$

Solving for $a$ gives the upper limit for stability, and any smaller value of $a$ is stable. Thus we obtain

$$a <= b(1+\sqrt{52}) \qquad \text{Q.E.D.}$$