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I was trying to show that such a positive integer could not exist and got as far as proving that if such a positive integer exists, it must be divisible by 9. Then my son, Tanawin Charoen-Rajapark, hoping to prove me wrong, succeeded and found one such integer. I trivially extend his result by finding all of the infinitely many such integers.

The smallest such integer has 18 digits: 105,263,157,894,736,842. Call this integer A and let f(x) be the function that maps one positive integer to another by moving its rightmost digit all the way to the left. Then f(A) = 210,526,315,789,473,684 = 2A as desired. There are eight such integers with 18 digits: they are, in increasing order, A, $f^{12}(A)$, f(A), $f^{15}(A)$, $f^{13}(A)$, $f^{5}(A)$, $f^{2}(A)$, and $f^{7}(A)$. We do not consider $f^{17}(A)$ valid since a leading zero should not be allowed.

Let g(x,k) be the function that concatenates k (positive) copies of a positive integer x together. For example, g(6789,3) = 678,967,896,789. Then if x is any of the above eight 18-digit integers, then for any positive integer k, g(x,k) also has the desired property that f(g(x,k)) = 2g(x,k). The concatenation thus generates infinitely many integers with the desired property.

The above discussion still leaves one wonder whether there may be other positive integers with the desired property. The answer is no, and we now explain. Let y be a positive integer with the desired property. The rightmost digit of y can neither be 0 nor 1. (If it is 0, then f(y) < y; and if it is 1, then the leftmost digit of f(y) is 1, forcing the leftmost digit of y to be at least 5 and 2y has one more digit than f(y).) Suppose the rightmost digit of y is 2. Then its next digit to the left must be 4 (= 2x2), followed (to the left) by 8 (= 4x2), 6 (= 8x2 – 10), 3 (= 6x2 – 10 + 1 (for the carry from 8x2)), and so on. Since the next digit to the left depends myopically on the current digit (for multiplication by 2) and the previous digit (for the addition of 1 for the carry, if any), this infinite sequence must repeat itself. When the rightmost digit of y is 2, this sequence generates A (shown above) before it repeats itself. The integer y is then of the form g(A,k) for some positive integer k, under the assumption that its rightmost digit is 2.

It turns out that when the rightmost digit of y is 3, 4, 5, 6, 7, 8, or 9, then the above iteration generates, respectively, $f^{12}(A)$, f(A), $f^{15}(A)$, $f^{13}(A)$, $f^{5}(A)$, $f^{2}(A)$, or $f^{7}(A)$, before the sequence repeats itself. Therefore, there can be no other integers with the desired property, apart from those that have been described.

Although there are infinitely many such integers, they are extremely rare, and so I was not that far off thinking they did not exist.

We may wonder what happens if the problem is generalized so that instead of requiring f(x) = 2x, we require f(x) = mx for some m = 3, 4, …, 9. (If m is 10 or more, such an x does not exist.) For a given m, let us refer to a positive integer x as an "m-seed" if x has the desired property (that is, f(x) = mx) and x is not g(y,k) for some positive integer y and some integer k > 1 (i.e., x is not a concatenation of two or more identical copies of some positive integer y). With this terminology, the eight 18-digit integers described above are thus the 2-seeds. Note that for a given m, if x is an m-seed, then for any positive integer k, g(x,k) has the desired property.

We now describe all the m-seeds for m = 3, 4, …, 9. They can be constructed by the same method as that used to construct A. For each m, it turns out that there are exactly 10 – m m-seeds.

m = 3: All seven 3-seeds have 28 digits. Let S_3 = 1,034,482,758,620,689,655,172,413,793. The seven 3-seeds are, in increasing order, S_3, $f^5(S_3)$, $f^9(S_3)$, $f^{17}(S_3)$, $f^7(S_3)$, $f^{22}(S_3)$, and $f(S_3)$.

m = 4: All six 4-seeds have 6 digits. They are 102,564; 128,205; 153,846; 179,487; 205,128 (= f^3(128,205)); and 230,769.

m = 5: This is the most interesting case. One 5-seed has 6 digits: the familiar 142,857. The four remaining 5-seeds, however, have 42 digits. Let S_5 = 102,040,816,326,530,612,244,897,959,183,673,469,387,755. The four remaining 5-seeds are, in increasing order, S_5, $f^{26}(S_5)$, $f^{35}(S_5)$, and $f^{15}(S_5)$.

m = 6: All four 6-seeds have 58 digits. Let S_6 = 1,016,949,152,542,372,881,355,932,203,389,830,508,474,576,271,186,440,677,966. The four 6-seeds are, in increasing order, S_6, $f^{13}(S_6)$, $f^{40}(S_6)$, and $f^{51}(S_6)$.

m = 7: All three 7-seeds have 22 digits. They are 1,014,492,753,623,188,405,797; 1,159,420,289,855,072,463,768; and 1,304,347,826,086,956,521,739.

m = 8: Both 8-seeds have 13 digits. They are 1,012,658,227,848; and 1,139,240,506,329.

m = 9: The 9-seed has 44 digits. It is 10,112,359,550,561,797,752,808,988,764,044,943,820,224,719.